Suspension of operads

Master's thesis in Mathematics

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Abstract

We study the suspension of operads, specifically in the ∞-categories of spectra and pointed spaces, arising in the literature in the context of Koszul duality. In the stable case, we investigate a conjectured characterising property of operadic suspension posed by Heuts–Land [HL24] and extend their positive result for the nonunital $\mathbb{E}_n$-operads to the $\mathbb{E}_\infty$-operad. We also discuss an alternative approach, to characterise operadic suspension in terms of invertibility with respect to the levelwise tensor product of operads. In the unstable case, we show that constructions of Arone–Kankaanrinta [AK14] and Ching–Salvatore [CS22] of a 'sphere operad' produce equivalent operads by point-set means. We briefly discuss to what extent the approaches for the stable case could generalise to the unstable case.

Contents

1 Introduction

This introduction is divided in four parts. The first two sections introduce two main concepts that feature in this thesis, operads and spectra, for a reader with some background in algebraic topology. The third outlines the research project and results, and describes the structure of the thesis. The final section briefly describes some notational conventions used throughout the thesis.

1.1 Operads and higher algebra

In this section, we give an introduction to the main object featuring in this thesis: operads. We discuss their historical origins in the study of iterated loop spaces and present the 'little cubes operads' constructed for this purpose. These operads are used to describe algebraic structures which are associative or commutative only up to homotopy. Finally, we discuss how ∞-categories can be incorporated into the story, which are also designed to deal with constructions that are defined up to homotopy.

Historically, operads were introduced around 1970 by Boardman–Vogt [BV68; BV73] and May [May72] to understand the structure of iterated loop spaces. Recall that for a pointed space X, the k-fold loop space $\Omega^k X$ is the space of all pointed maps $S^k \to X$. Its set of path components is the kth homotopy group of X, denoted $\pi_k X$. The homotopy groups of a pointed space inherit a group structure (which is abelian for $k > 1$) from the composition of loops in $\Omega^k X$. Thus, the algebraic structure on homotopy groups comes from some sort of algebraic structure on loop spaces. However, composition of loops is not associative on the nose, but only up to homotopy; and these homotopies are again structured in some coherent way, again up to homotopy. It turns out that the homotopy-coherent algebraic structure carried by iterated loop spaces is fully described by certain *operads*, the little k-cubes operads $\mathbb{E}_k$. In fact, under mild hypotheses, a pointed space admitting the structure of an 'algebra' over such an operad has the homotopy type of an iterated loop space.

A *(topological) operad* $\mathbf{O}$ consists of a space $\mathbf{O}(n)$ for every $n \geqslant 0$, whose elements we think of as 'operations' with n inputs and one output. For all n and $k_1, \dots, k_n$, there is a *composition* map

$$\mathbf{O}(n) \times \mathbf{O}(k_1) \times \cdots \times \mathbf{O}(k_n) \to \mathbf{O}(k_1 + \cdots + k_n),$$

sending an operation $o \in \mathbf{O}(n)$ and operations $p_i \in \mathbf{O}(k_i)$ to a composed operation $o \circ (p_1, \dots, p_n)$ with $k_1 + \cdots + k_n$ inputs. Intuitively, one uses the output of the ith operation p_i as the ith input of o. The composition map should be continuous, and it should satisfy some associativity and permutation conditions (see e.g. [HM22]).

The idea is to interpret these abstract operations as actual n-ary operations $X^n \to X$ on a concrete space X, exhibiting X as an 'algebra' over the operad $\mathbf{O}$. To every operation $o \in \mathbf{O}(n)$, we associate a map $X^n \to X$, in such a way as to respect the structure of the operad $\mathbf{O}$; for instance, composition of operations in $\mathbf{O}$ should correspond to composition of functions.

One of the simplest operads is the *commutative operad* Com, which is given by the one-point space $\mathrm{Com}(n) := *$ for all n. An algebra over the commutative operad is a coherent choice of a single n-ary operation $X^n \to X$ on a space X for every n. The coherence encodes that algebras over the commutative operad are precisely commutative topological monoids. The next simplest operad is the *associative operad* Assoc whose nth term $\mathrm{Assoc}(n) := \Sigma_n$ is the symmetric group on n letters. Composition in the associative operad is given by composing permutations in a certain natural way. Algebras over the associative operad are precisely topological monoids (not necessarily commutative).

As we discussed above, the multiplication operation of a loop space ΩX is not associative on the nose. The structure of the associative operad is thus too strict to describe its algebraic structure. A more 'flexible' or more homotopical variant of the associative operad is the 'little intervals operad' $\mathbb{E}_1$. This operad consists of spaces $\mathbb{E}_1(n)$ whose points represent a choice of n 'little' subintervals of the standard interval $[0, 1]$; the interiors of these subintervals may not overlap. For instance, a point of $\mathbb{E}_1(2)$ (a binary operation) consists of a choice of two subintervals $[a_1, b_1], [a_2, b_2] \subseteq [0, 1]$ such that $(a_1, b_1) \cap (a_2, b_2) = \varnothing$. The space $\mathbb{E}_1(n)$ can be topologised as a subspace of $\mathbb{R}^{2n}$. The composition operation of the operad $\mathbb{E}_1$ is defined by 'shrinking little intervals into other little intervals', as pictured in Figure 1.

The $\mathbb{E}_1$-operad naturally acts on loop spaces ΩX: given n little intervals $[a_i, b_i]$ and n loops w_i in X, we define a composed loop $[0, 1] \to X$ that runs through w_i on the

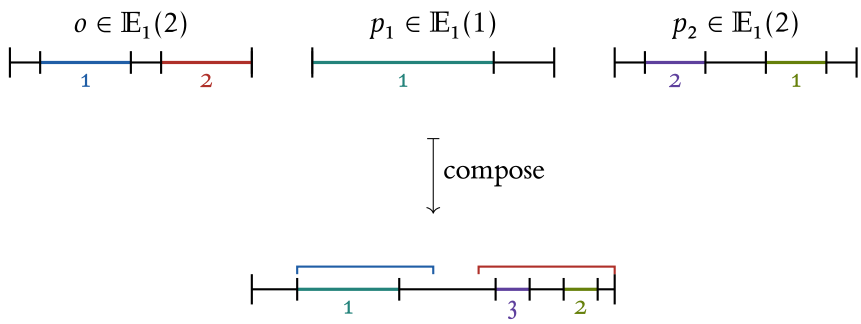
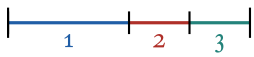


Figure 1 Composition in the little intervals operad $\mathbb{E}_1$

little interval $[a_i, b_i]$ and stands still at the basepoint of X outside the little intervals. The requirement that the interiors of the little intervals do not overlap ensures that this composition is well-defined. This action of $\mathbb{E}_1$ on a loop space ΩX describes the associativity of composition of loops: for three loops u, v, w in X, the composites $u(vw)$ and $(uv)w$ are not equal on the nose, but there is a path in the space $\mathbb{E}_1(3)$ between the two parametrisations



and

of these composites, passing to a homotopy between the paths $u(vw)$ and $(uv)w$ under the $\mathbb{E}_1$-action.

The monoid structure on the fundamental group $\pi_1 X = \pi_0(\Omega X)$ (that is, the associative multiplication) is now a consequence of the algebraic structure of the $\mathbb{E}_1$-operad. In fact, applying the path components functor π_0 levelwise to $\mathbb{E}_1$, we obtain the associative operad Assoc in the category of sets (since up to homotopy, only the order of the little intervals matters, and choosing an order is just choosing a permutation). Thus, the $\mathbb{E}_1$-algebra structure on ΩX gives an Assoc-algebra structure on $\pi_0(\Omega X)$, which is nothing more than the structure of a monoid. (We also know that $\pi_1 X$ is a *group*, i.e. that the multiplication has inverses, but this information is *not* encoded by the $\mathbb{E}_1$-operad. Instead, we say that ΩX is a *grouplike* $\mathbb{E}_1$-algebra.) In this sense, the $\mathbb{E}_1$-operad is a 'homotopical variant' of the associative operad: its algebras are not strict topological monoids, but topological spaces with a binary operation that is associative up to coherent homotopy.

More generally, one can define a 'little k-cubes operad' $\mathbb{E}_k$ for any positive integer k. For $k > 1$, one replaces the interval of the $\mathbb{E}_1$-operad by the k-dimensional cube $[0, 1]^k$. Where the $\mathbb{E}_1$-operad acts on all loop spaces, the $\mathbb{E}_k$-operad acts on k-fold spaces $\Omega^k X$. The commutativity of $\pi_k X$ for $k > 1$ (or the noncommutativity of $\pi_1 X$) can now be seen as a homotopical commutativity (or noncommutativity) of the $\mathbb{E}_k$-operads themselves: in the 'little squares operad' $\mathbb{E}_2$ for instance, we can move two little squares around each other in a continuous manner without them overlapping (see Figure 2), but we cannot make two little intervals in the little intervals operad $\mathbb{E}_1$ swap places continuously.

There are natural inclusion maps $\mathbb{E}_k \hookrightarrow \mathbb{E}_{k+1}$ which induce restriction maps between the corresponding categories of algebras: any $\mathbb{E}_{k+1}$-algebra is also naturally an $\mathbb{E}_k$-algebra. One can see the algebras of the $\mathbb{E}_k$-operads as increasingly more commutative, moving along the sequence

$$\mathbb{E}_1 \hookrightarrow \mathbb{E}_2 \hookrightarrow \mathbb{E}_3 \hookrightarrow \dots .$$

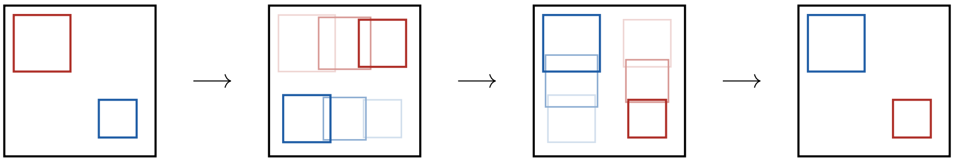


Figure 2 Continuously interchanging two little squares, describing the commutativity of the $\mathbb{E}_2$-operad

We can take this even further by taking the colimit of this sequence, by which we obtain the $\mathbb{E}_\infty$*-operad*. The algebras of the $\mathbb{E}_\infty$-operad are highly commutative and possess a lot of structure.

The result mentioned earlier, that the $\mathbb{E}_k$-operad fully describes the algebraic structure of k-fold loop spaces, can now be made more precise:

Theorem *If a connected pointed space Y admits a grouplike algebra structure of the $\mathbb{E}_k$-operad, then it is weakly equivalent to a k-fold loop space $\Omega^k X$.*

Iterated loop spaces are thus characterised up to homotopy by whether they admit a (grouplike) action of the $\mathbb{E}_k$-operad. This result is due to Boardman–Vogt [BV73] in the case $k = 1$ and to May [May72] for all $k \geqslant 1$. A higher-categorical version can be found in Lurie's [DAG-VI, Theorem 1.3.6]. Passing to the colimit, the grouplike algebras of the $\mathbb{E}_\infty$-operad are precisely 'infinite loop spaces'; this was proven in [BV68].

We have already seen above that the $\mathbb{E}_1$-operad is a 'homotopical variant' of the associative operad, at least at the level of algebras. The work of encoding the homotopical coherence was in the construction of the operad, but encoding such coherence is also precisely the goal of *∞-categories* (more precisely, *(∞,1)-categories*); roughly, these are categories where composition is defined only up to homotopy. Indeed, one can define operads (then also called '∞-operads') in the ∞-category of spaces, and these operads fit together into their own ∞-category. The study of ∞-operads and their algebras is known as 'higher algebra'. If one considers the associative operad and the $\mathbb{E}_1$-operad in this setting, it turns out they are equivalent as objects of the ∞-category of operads: there is a map between them which is invertible up to homotopy (this notion is the ∞-categorical generalisation of objects of 1-categories being isomorphic). Thus, for all homotopy-coherent purposes, one need not distinguish these operads: that the binary operation of an associative algebra in the ∞-category of spaces is only associative up to homotopy is now already encoded by the ∞-categorical structure.

A similar story can be told for the $\mathbb{E}_\infty$-operad: in the ∞-categorical setting, it is equivalent to the commutative operad. The situation can be summed up by the following diagram:

$$\mathrm{Assoc} \simeq \mathbb{E}_1 \hookrightarrow \mathbb{E}_2 \hookrightarrow \mathbb{E}_3 \hookrightarrow \ldots \hookrightarrow \mathbb{E}_\infty \simeq \mathrm{Com}.$$

The $\mathbb{E}_k$-algebras thus interpolate between associative and commutative algebras. At the two ends of this sequence, the passage from 1-categories to ∞-categories allows us to use simpler constructions for the same purposes.

1.2 Spectra and stable homotopy theory

Another category that will prominently feature in this thesis is the ∞-category Sp of *spectra*, which is the foundation of *stable homotopy theory*. The fundamental motivation for stable homotopy theory is the Freudenthal suspension theorem, which says that for a pointed space X satisfying some assumptions, the sequence

$$\pi_k X \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \ldots \to \pi_{k+n}(\Sigma^n X) \to \ldots$$

stabilises for sufficiently large n. In other words, the homotopy groups of X stabilise after suspending X sufficiently many times. The colimit of the sequence is called the *kth stable homotopy group* of X.

The space X and its iterated suspensions give rise to a spectrum, which is a sequence of pointed spaces $E(n)$ for all $n \geqslant 0$, equipped with equivalences $E(n) \simeq \Omega E(n+1)$; applying the free infinite loop space functor to X and its suspensions produces such a spectrum. The stable homotopy groups of X can then be seen as the homotopy groups of the resulting *suspension spectrum* $\Sigma^\infty X$. In particular, for the 0-sphere $X = S^0$, the suspension spectrum $\mathbb{S} := \Sigma^\infty S^0$ is known as the *sphere spectrum*, and its homotopy groups are the stable homotopy groups of spheres.

One way to think about the ∞-category of spectra, which ties in to our earlier discussion of the algebraic structure possessed by loop spaces, is as an 'algebraic' variant of the ∞-category of spaces (following [HA, § 1.4, perspective (D)]). Given any spectrum E, the space $E(0)$ is equivalent to an n-fold loop space $\Omega^n E(n)$ for every n; in other words, it is an infinite loop space. Therefore, it admits an action of the $\mathbb{E}_\infty$-operad, or equivalently, it is a commutative monoid in the ∞-category Spc of spaces. In fact, there is an equivalence between the full subcategory of *connective* spectra and the grouplike $\mathbb{E}_\infty$-algebras in Spc (see [HA, § 5.2.6]). This viewpoint allows us to think of spectra as 'algebraic spaces'; the relation between the ∞-category of spectra and the ∞-category of spaces is roughly analogous to the relation between the category of abelian groups and the category of sets (in the latter category, the grouplike commutative monoids are indeed precisely abelian groups).

Another way to think of the ∞-category of spectra is as the *stabilisation* of the ∞-category Spc_* of pointed spaces. We refer to [HA, § 1.1.1] for the definition of stable ∞-categories, but for our current purposes it suffices to know the following about them. A stable ∞-category is in particular pointed, which means it has a zero object 0 (i.e., an object that is both initial and terminal). In a pointed ∞-category $\mathscr{C}$ with pullbacks and pushouts, one can define the suspension and loop functors $\Sigma : \mathscr{C} \to \mathscr{C}$ and $\Omega : \mathscr{C} \to \mathscr{C}$: the suspension of $X \in \mathscr{C}$ is the pushout of the map $X \to 0$ along itself (dually, one defines the loop functor via pullbacks). In general, there is an adjunction $\Sigma \dashv \Omega$, but in a stable ∞-category, these functors are moreover equivalences. Roughly, the ∞-category of spectra is characterised by the fact that it is universal among stable ∞-categories equipped with a forgetful functor $\mathrm{Sp} \to \mathrm{Spc}_*$ (a precise statement is [HA, Corollary 1.4.2.23]).

1.3 Summary and outline

In this thesis, we study the operation of *operadic suspension* on the ∞-category of operads, specifically in pointed spaces or spectra. To an operad $\mathbf{O}$, one can associate a 'suspended' operad $\mathbf{sO}$, whose algebras satisfy the following property: suspending the underlying object, an $\mathbf{sO}$-algebra structure on an object X induces an $\mathbf{O}$-algebra structure on ΣX. Moreover, this assignment is an equivalence in the stable case on account of suspension being an equivalence. The terms of the suspended operad $\mathbf{sO}$ are given by iterated suspensions

$$\mathbf{sO}(n) \simeq \Sigma^{n-1}\mathbf{O}$$

of those of $\mathbf{O}$, with appropriate Σ_n-actions and structure maps. Ching–Salvatore [CS22] use operadic suspension in the setting of Koszul duality, and show that the *Koszul dual* $K\mathbb{E}_k$ of the $\mathbb{E}_k$-operad in spectra is equivalent to the k-fold desuspension $\mathbf{s}^{-k}\mathbb{E}_k$. In the setting of higher algebra, operadic suspension is used by Heuts–Land [HL24] and Antolín-Camerena–Brantner–Heuts [ABH25] to prove relations between different $\mathbb{E}_k$-operads and their algebras. Similar results had already been obtained in the R-linear case (for R an ordinary commutative ring) by Berger–Fresse [BF04] and by Fresse [Fre11].

The construction of operadic suspension naturally generalises from the R-linear case to the stable case, that is, to operads in the ∞-category in spectra. In the unstable case, however, the situation is more complicated. However, Arone–Kankaanrinta [AK14] and Ching–Salvatore [CS22] construct so-called 'sphere operads' $\mathbf{S}$ in the category of pointed spaces, with $\mathbf{S}(n) \simeq S^{n-1}$. Using such a sphere operad, one can define operadic suspension of operads in pointed spaces by levelwise smashing with $\mathbf{S}$. Applying the suspension spectrum functor levelwise then recovers the operadic suspension in the stable case; in particular, the two models are *stably* equivalent.

The constructions of Arone–Kankaanrinta and Ching–Salvatore are not isomorphic as topological operads: the terms of the former sphere operad are homeomorphic to spheres, but those of the latter are only homotopy equivalent to spheres. Given the existence of multiple different constructions of sphere operads and their corresponding operadic suspension, it would be desirable to characterise these operads by universal properties. One would then obtain a comparison between the different constructions by showing that both models satisfy the universal properties.

The goal of this research project is to characterise operadic suspension, taking up a question posed by Heuts–Land in [HL24, § 3.5].

We start with the stable case, which is in many aspects simpler, essentially due to the fact that the suspension and loop functors are equivalences. We discuss two approaches to a characterisation of stable operadic suspension. The first approach, suggested by Heuts–Land, is phrased in terms of a property of a suspension morphism $\sigma : \mathbf{O} \to \mathbf{sO}$; a similar approach is employed by Fresse [Fre11] in the linear case. We extend the result that the $\mathbb{E}_k$-operads (for finite k) satisfy this property, as established by Heuts–Land, to the $\mathbb{E}_\infty$-operad in Proposition 3.5. We discuss to what extent the proposed property characterises operadic suspension, and we discuss additional assumptions one can impose to determine the suspension morphism up to equivalence (see Proposition 3.9).

The second approach is to characterise operadic suspension in spectra via invertibility with respect to the levelwise tensor product. An object X of a symmetric monoidal ∞-category is invertible if there is an object Y such that $X \otimes Y$ is equivalent to the monoidal unit. For the ∞-category of nonunital operads in spectra, we conjecture in Conjecture 3.14 that the invertible objects are precisely the suspensions of the $\mathbb{E}_\infty$-operad (the monoidal unit):

Conjecture *A nonunital operad in the symmetric monoidal ∞-category of spectra is invertible with respect to the levelwise tensor product if and only if it is equivalent to a suspension $\mathbf{s}^d\mathbb{E}_\infty$ of the $\mathbb{E}_\infty$-operad for some $d \in \mathbb{Z}$.*

This statement is analogous to the category of spectra itself, where the invertible objects are precisely the suspensions $\mathbb{S}^d = \Sigma^d\mathbb{S}$ of the sphere spectrum. We discuss how the condition of invertibility for operads relates to a weaker condition, which is easier to check in practice:

we say an operad $\mathbf{O}$ is *quasi-invertible* if the terms $\mathbf{O}(n)$ are invertible for all n and if all partial composition maps are equivalences. In Proposition 3.18, we prove that invertible operads are quasi-invertible, and we expect the converse to be true as well. We reduce the conjecture about the Picard group to a characterisation of the nonunital $\mathbb{E}_\infty$-operad as the essentially unique operad structure on its underlying symmetric sequence with partial composition maps being equivalences (see Remark 3.21).

In the unstable case, we discuss the constructions of Arone–Kankaanrinta and Ching–Salvatore of a sphere operad, and prove in Theorem 4.5 by point-set means that these are equivalent:

Theorem *The sphere operads of Arone–Kankaanrinta and Ching–Salvatore are weakly equivalent.*

Finally, we discuss to what extent the approaches for a characterisation of operadic suspension in the stable case generalise to the unstable case.

Outline In § 2, we define the ∞-category of operads in a symmetric monoidal ∞-category. We discuss the monad associated to an operad and define the ∞-category of algebras over an operad as the category of algebras over this monad. We also present some examples of operads, notably the $\mathbb{E}_n$-operads and the $\mathbb{E}_\infty$-operad.

In § 3, we define operadic suspension in the stable case. We then study the two approaches to a characterisation of operadic suspension, via the suspension morphism in § 3.2 and via invertibility in § 3.3.

In § 4, we indicate how to define operadic suspension unstably via a sphere operad. We prove in § 4.1 that the constructions of a sphere operad of [AK14] and [CS22] are equivalent. In § 4.2 we discuss the suitability of the approaches for stable operadic suspension in the unstable case.

1.4 Conventions

Throughout, we work in the setting of ∞-categories as developed by Lurie [HTT; HA], and we do not notationally distinguish a 1-category and its nerve. Our use of the prefix '∞-' is not consistent, and the word 'category' in general refers to an ∞-category. We implicitly use quasicategories as our model for ∞-categories; except for references to the literature, however, we will not have to depend on their specific characteristics. For the theory of ∞-categories and (symmetric) monoidal ∞-categories, we refer to [HTT; HA].

We write:

- Spc for the ∞-category of spaces of [HTT, Definition 1.2.16.1]. The ∞-category of spaces is characterised by the fact that it is the free cocomplete ∞-category generated by a single object, the one-point space (see [HTT, Theorem 5.1.5.6]). It is equipped with the cartesian symmetric monoidal structure.

- Spc_* for the ∞-category of pointed spaces. The smash product $\wedge$ equips the ∞-category of pointed spaces with a symmetric monoidal structure, which is essentially uniquely determined by the requirements that the monoidal product preserves colimits in each variable separately and that the 0-sphere S^0 is the monoidal unit (see [HA, Remark 4.8.2.14]).

- Sp for the ∞-category of spectra. This category is characterised by the fact that it is the free stable and cocomplete ∞-category generated by a single object, the sphere spectrum $\mathbb{S}$ (see [HA, Corollary 1.4.4.6]) The smash product $\otimes$ of spectra equips Sp with a symmetric monoidal structure, which is essentially uniquely determined by the requirements that the monoidal product preserves colimits in each variable separately and that the sphere spectrum is the monoidal unit (see [HA, Corollary 4.8.2.19]).

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2 Operads

In this section, we present a definition of the ∞-category of operads in a symmetric monoidal ∞-category, also called 'enriched ∞-operads' in the literature. Enriched ∞-operads are a generalisation of (unenriched) ∞-operads, which can then be seen as ∞-operads enriched in the cartesian symmetric monoidal ∞-category of spaces. There are multiple models of ∞-operads in the literature; first and best developed are the ∞-operads of Lurie [HA], but there are also the dendroidal sets of Moerdijk–Weiss [MW07], the dendroidal Segal spaces of Cisinski–Moerdijk [CM13] and the complete Segal operads of Barwick [Bar18]. All these models of ∞-operads are equivalent by results of [HHM16; Bar18; CHH18].

The development of models for enriched ∞-operads is more recent. Following [HL24] and [ABH25], we define operads as algebras in the category of symmetric sequences with respect to the monoidal structure given by the composition product. This generalises the well-known definition of operads in symmetric monoidal 1-categories via symmetric sequences due to Kelly [Kel05]. In the ∞-categorical setting, the composition product is constructed from universal properties by Brantner–Campos–Nuiten [BCN24], following the 1-categorical argument due to Carboni and presented by Kelly [Kel05] and Trimble [Tri]. Haugseng [Hau22] defines enriched ∞-operads via the composition product by a different method. There are other approaches to defining enriched ∞-operads, such as the approach of Chu–Haugseng [CH20] based on Barwick's complete Segal operads definition of 'unenriched' ∞-operads; a comparison between this model and the model via symmetric sequences of [BCN24] is currently not available in the literature.

For the discussion of the ∞-category of operads, we follow [Bra17, § 4.1.2], [BCN24, § 3.1], and [ABH25, § 2]. Throughout, we will let $\mathscr{C}$ denote a presentable symmetric monoidal ∞-category (which is thus in particular complete and cocomplete), and we assume that the tensor product commutes with colimits in both variables separately. We write $\otimes$ for the tensor product of $\mathscr{C}$ and $\mathbb{1}$ for the monoidal unit. In particular, $\mathscr{C}$ might be the symmetric monoidal ∞-category of pointed spaces or spectra, the derived category of a ring, or a symmetric monoidal 1-category satisfying the same assumptions. The assumption that the tensor product commutes with colimits is satisfied in particular if the monoidal structure is closed, i.e., if the functor $X \otimes - : \mathcal{C} \to \mathcal{C}$ has a right adjoint for every object X.

At the end of the section, we give some examples of ∞-operads, most importantly the $\mathbb{E}_\infty$-operad and the $\mathbb{E}_n$-operads, which will play an important role in the next section.

Symmetric sequences We write $\mathrm{Fin}^\simeq$ for the groupoid of finite sets and bijections.

Definition 2.1 Let $\mathcal{C}$ be an ∞-category. The ∞-category of *symmetric sequences* in $\mathcal{C}$ is the functor category

$$\mathrm{SSeq}(\mathcal{C}) := \mathrm{Fun}(\mathrm{Fin}^\simeq, \mathcal{C}).$$

For a symmetric sequence A in $\mathcal{C}$, we write $A(n)$ for the value of A on a set with n elements. Functoriality equips the object $A(n)$ of $\mathcal{C}$ with an action of the symmetric group Σ_n.

There is a natural inclusion $\iota : \mathcal{C} \hookrightarrow \mathrm{SSeq}(\mathcal{C})$ sending an object X to the symmetric sequence concentrated in level zero with value X.

Day convolution The monoidal structure on $\mathrm{Fin}^\simeq$ given by disjoint union induces the symmetric monoidal structure of *Day convolution* on the category of symmetric sequences (cf. [HA, Corollary 4.8.1.12]). It is given by

$$(A \otimes B)(n) = \bigoplus_{a+b=n} (\Sigma_n \otimes A(a) \otimes B(b))_{h(\Sigma_a \times \Sigma_b)},$$

where Σ_n denotes the Σ_n-induction of the monoidal unit $\mathbb{1}$, and where $\Sigma_a \times \Sigma_b \hookrightarrow \Sigma_n$ denotes the obvious subgroup.

The inclusion $\iota : \mathcal{C} \hookrightarrow \mathrm{SSeq}(\mathcal{C})$ is symmetric monoidal for the Day convolution product, providing a tensoring of $\mathrm{SSeq}(\mathcal{C})$ over $\mathcal{C}$.

Composition product Another monoidal structure on the ∞-category of symmetric sequences is the *composition product*, constructed in [BCN24, § 3.1]. For symmetric sequences A and B, it is given by

$$A \circ B = \bigoplus_{n \geqslant 0} (A(n) \otimes B^{\otimes n})_{h\Sigma_n},$$

where $B^{\otimes n}$ is equipped with a Σ_n-action by permutation of the tensor product factors. The monoidal unit with respect to the composition product is the symmetric sequence concentrated in degree one with value $\mathbb{1}$. The composition product preserves all colimits on the left and it preserves sifted colimits and finite sifted limits on the right [BCN24, Remark 3.5].

For a symmetric sequence A, the restriction $\mathcal{C} \to \mathrm{SSeq}(\mathcal{C})$, $X \mapsto A \circ \iota(X)$ of the composition product along ι on the right takes values in the essential image of ι. Indeed, since ι is symmetric monoidal with respect to Day convolution, we have $\iota(X)^{\otimes n} \simeq \iota(X^{\otimes n})$ for any object X of $\mathcal{C}$, so it follows from the formulas that $A \circ \iota(X)$ is concentrated in degree zero. Thus, the restriction factors through ι via an endofunctor free_A on $\mathcal{C}$, given by

$$\mathrm{free}_A(X) = \bigoplus_{n \geqslant 0} (A(n) \otimes X^{\otimes n})_{h\Sigma_n},$$

and this defines a monoidal functor $\mathrm{free} : \mathrm{SSeq}(\mathcal{C}) \to \mathrm{End}(\mathcal{C})$ to the monoidal category of endofunctors under composition, since it is the restriction of the composition product to the essential image of ι.

Definition 2.2 The ∞-category of *operads* in $\mathscr{C}$ is defined to be the ∞-category

$$\mathrm{Opd}(\mathscr{C}) := \mathrm{Alg}(\mathrm{SSeq}(\mathscr{C}))$$

of algebra objects in the monoidal ∞-category $(\mathrm{SSeq}(\mathscr{C}), \circ)$.

Remark 2.3 If $\mathbf{O}$ is a symmetric sequence in a symmetric monoidal 1-category, then an algebra structure on $\mathbf{O}$ with respect to the composition product consists of a multiplication map $\mathbf{O} \circ \mathbf{O} \to \mathbf{O}$ and a unit $\mathbb{1} \to \mathbf{O}$ (together with some commutative diagrams). Expanding the formulas, we see that

$$(\mathbf{O} \circ \mathbf{O})(k) \simeq \coprod_{n \geqslant 0} \mathbf{O}(n) \otimes_{\Sigma_n} \Big(\coprod_{k_1 + \cdots + k_n = k} (\Sigma_k \otimes \mathbf{O}(k_1) \otimes \ldots \otimes \mathbf{O}(k_n))_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_n}} \Big),$$

and a map of symmetric sequences $\mathbf{O} \circ \mathbf{O} \to \mathbf{O}$ consists precisely of equivariant maps

$$\mathbf{O}(n) \otimes \mathbf{O}(k_1) \otimes \ldots \otimes \mathbf{O}(k_n) \to \mathbf{O}(k_1 + \cdots + k_n)$$

specifying the usual operadic composition maps. Thus, the above definition recovers the usual 1-category of operads for symmetric monoidal 1-categories.

As in the 1-categorical case, an operad $\mathbf{O}$ in a symmetric monoidal ∞-category comes equipped with composition maps

$$\circ : \mathbf{O}(n) \otimes \mathbf{O}(k_1) \otimes \ldots \otimes \mathbf{O}(k_n) \to \mathbf{O}(k_1 + \cdots + k_n).$$

These satisfy equivariance and compatibility conditions, but expressed by a large amount of coherence data. For $1 \leqslant i \leqslant n$, we denote the ith partial composition map of an operad $\mathbf{O}$ by

$$\circ_i : \mathbf{O}(n) \otimes \mathbf{O}(k) \to \mathbf{O}(n + k - 1).$$

Remark 2.4 In what follows, all operads $\mathbf{O}$ are assumed to be *nonunital*: they satisfy $\mathbf{O}(0) = *$. Correspondingly, we write $\mathrm{Opd}(\mathscr{C})$ for the category of nonunital operads. When we want to be explicit, we also write $\mathrm{Opd}^{\mathrm{nu}}(\mathscr{C})$ to stress nonunitality.

Associated monads Recall that the ∞-category of algebra objects in the monoidal ∞-category $(\mathrm{End}(\mathscr{C}), \circ)$ of endofunctors under composition is the ∞-category $\mathrm{Mnd}(\mathscr{C})$ of monads on $\mathscr{C}$. The monoidal functor $\mathrm{free} : \mathrm{SSeq}(\mathscr{C}) \to \mathrm{End}(\mathscr{C})$ thus induces a functor $\mathrm{free} : \mathrm{Opd}(\mathscr{C}) \to \mathrm{Mnd}(\mathscr{C})$ commuting with the forgetful functors as in the diagram

$$\begin{array}{ccc} \mathrm{Opd}(\mathscr{C}) & \xrightarrow{\mathrm{free}} & \mathrm{Mnd}(\mathscr{C}) \\ {\scriptstyle \mathrm{fgt}}\downarrow & & \downarrow{\scriptstyle \mathrm{fgt}} \\ \mathrm{SSeq}(\mathscr{C}) & \xrightarrow[\mathrm{free}]{} & \mathrm{End}(\mathscr{C}) \end{array}$$

For an operad $\mathbf{O}$, we call $\mathrm{free}_{\mathbf{O}}$ the *associated monad* of $\mathbf{O}$. If $\mathbf{O}$ is a nonunital operad, its associated monad is a reduced functor, that is, it satisfies $\mathrm{free}_{\mathbf{O}}(*) \simeq *$.

We define the category of algebras over an operad $\mathbf{O}$ as the category of algebras over the associated monad $\mathrm{free}_{\mathbf{O}}$.

Definition 2.5 For an operad $\mathbf{O}$ in $\mathscr{C}$, the ∞-category $\mathrm{Alg}_{\mathbf{O}}(\mathscr{C})$ of $\mathbf{O}$-*algebras* is the ∞-category of algebras over the associated monad $\mathrm{free}_{\mathbf{O}}$:

$$\mathrm{Alg}_{\mathbf{O}}(\mathscr{C}) := \mathrm{Alg}_{\mathrm{free}_{\mathbf{O}}}(\mathscr{C}).$$

Levelwise tensor product The third monoidal structure on $\mathrm{SSeq}(\mathscr{C})$ that we consider is the *levelwise tensor product* $\otimes_{\mathrm{lev}}$ given by

$$(A \otimes_{\mathrm{lev}} B)(n) := A(n) \otimes B(n).$$

We will use the levelwise tensor product in § 3 to define operadic suspension. It is shown in [BCN24, Proposition 3.9] that the functor $\otimes_{\mathrm{lev}} : \mathrm{SSeq}(\mathscr{C}) \times \mathrm{SSeq}(\mathscr{C}) \to \mathrm{SSeq}(\mathscr{C})$ has a lax monoidal structure with respect to the composition product, so it induces a functor $\otimes_{\mathrm{lev}} : \mathrm{Opd}(\mathscr{C}) \times \mathrm{Opd}(\mathscr{C}) \to \mathrm{Opd}(\mathscr{C})$ at the level of operads; in particular, for operads $\mathbf{O}$ and $\mathbf{P}$, the levelwise tensor product $\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P}$ is also an operad. Correspondingly, there is a natural functor

$$\mathrm{Alg}_{\mathbf{O}}(\mathscr{C}) \times \mathrm{Alg}_{\mathbf{P}}(\mathscr{C}) \to \mathrm{Alg}_{\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P}}(\mathscr{C}), \quad (A, B) \mapsto A \otimes B. \tag{1}$$

Associated monads and colimits In our discussion of operadic suspension in the next sections, we will need a result about the preservation of colimits by the functor $\mathrm{free} : \mathrm{Opd}(\mathscr{C}) \to \mathrm{Mnd}(\mathscr{C})$, which we briefly discuss here. This functor is induced by a monoidal functor from symmetric sequences in $\mathscr{C}$ to endofunctors on $\mathscr{C}$. For any symmetric sequence A, the endofunctor $\mathrm{free}_A \in \mathrm{End}(\mathscr{C})$ is given by colimits and tensor powers, both of which preserve sifted colimits (see [HTT, Proposition 5.5.8.6] for the latter), so it preserves sifted colimits. Moreover, by the same observation and since colimits are computed levelwise in $\mathrm{SSeq}(\mathscr{C})$ and $\mathrm{End}(\mathscr{C})$ (they are functor categories with domains $\mathrm{Fin}^{\simeq}$ and $\mathscr{C}$), the functor $\mathrm{free} : \mathrm{SSeq}(\mathscr{C}) \to \mathrm{End}(\mathscr{C})$ preserves sifted colimits. We will need the following lift of this result to the level of categories of algebras.

Lemma 2.6 *The functor* $\mathrm{free} : \mathrm{Opd}(\mathscr{C}) \to \mathrm{Mnd}(\mathscr{C})$ *preserves sifted colimits.*

We will apply this result to the associated monad of the $\mathbb{E}_\infty$-operad, which is a sifted (and *a fortiori* filtered) colimit of the $\mathbb{E}_n$-operads, in the proof of Proposition 3.5.

Remark 2.7 In the proof of [Heu24, Theorem 5.2], it is shown that $\mathrm{free} : \mathrm{Opd}(\mathscr{C}) \to \mathrm{Mnd}(\mathscr{C})$ preserves all colimits, after first showing it preserves sifted colimits, which is more straightforward. Sifted colimits suffice for our current purposes, so we only discuss this argument.

Proof (of Lemma 2.6) Write $\mathrm{End}^{\Sigma}(\mathscr{C})$ for the full monoidal subcategory of $\mathrm{End}(\mathscr{C})$ spanned by the functors that preserve sifted colimits, and write $\mathrm{Mnd}^{\Sigma}(\mathscr{C})$ for the corresponding category of monads, which contains the essential image of $\mathrm{free} : \mathrm{Opd}(\mathscr{C}) \to \mathrm{Mnd}(\mathscr{C})$. In the commutative square

$$\begin{array}{ccc} \mathrm{Opd}(\mathscr{C}) & \xrightarrow{\mathrm{free}} & \mathrm{Mnd}^{\Sigma}(\mathscr{C}) \\ {\scriptstyle \mathrm{fgt}}\downarrow & & \downarrow{\scriptstyle \mathrm{fgt}} \\ \mathrm{SSeq}(\mathscr{C}) & \xrightarrow[\mathrm{free}]{} & \mathrm{End}^{\Sigma}(\mathscr{C}) \end{array}$$

the vertical forgetful functors create sifted colimits by [HA, Proposition 3.2.3.1] and the bottom horizontal functor preserves them, implying the desired conclusion. □

Examples of operads In this section, we briefly describe some examples of operads, specifically the commutative and associative operads, the $\mathbb{E}_n$-operads for finite n, and the $\mathbb{E}_\infty$-operad. We will discuss these operads in the fundamental case of the cartesian symmetric monoidal ∞-category Spc of spaces, as this will automatically extend to other symmetric monoidal ∞-categories by the following reasons.

First, the ∞-category of spaces is 'freely generated under colimits by the one-point space': for any presentable ∞-category $\mathscr{C}$, evaluating at the one-point space induces an equivalence

$$\mathrm{Fun}^L(\mathrm{Spc}, \mathscr{C}) \xrightarrow{\simeq} \mathscr{C}$$

between the ∞-category of colimit-preserving functors $\mathrm{Spc} \to \mathscr{C}$ and $\mathscr{C}$ (see [HTT, Theorem 5.1.5.6 and Corollary 5.5.2.9]). In other words, a colimit-preserving functor $\mathrm{Spc} \to \mathscr{C}$ is determined by its value on the one-point space, and conversely, any object $X \in \mathscr{C}$ gives rise to a colimit-preserving functor sending the one-point space to X.

Second, we can upgrade this result to the context of symmetric monoidal ∞-categories. The cartesian symmetric monoidal structure is the essentially unique symmetric monoidal structure on the ∞-category of spaces with the one-point space as monoidal unit and such that the monoidal product preserves colimits in each variable separately (see [HA, Corollary 4.8.1.12]). Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a presentable symmetric monoidal ∞-category and assume the tensor product preserves colimits in each variable separately. Using the equivalence above, a colimit-preserving symmetric monoidal functor $\mathrm{Spc} \to \mathscr{C}$ is determined by its value at the one-point space, which is the monoidal unit of Spc and must thus be sent to the monoidal unit $\mathbb{1}$ of $\mathscr{C}$. Conversely, sending the one-point space to $\mathbb{1}$ defines a colimit-preserving symmetric monoidal functor, and this is thus the essentially unique such functor. This functor induces a colimit-preserving monoidal functor $\mathrm{SSeq}(\mathrm{Spc}) \to \mathrm{SSeq}(\mathscr{C})$ at the level of symmetric sequences, which in turn induces a functor $\mathrm{Opd}(\mathrm{Spc}) \to \mathrm{Opd}(\mathscr{C})$ at the level of operads. Using this functor, we can pass operads in spaces to other symmetric monoidal ∞-categories.

Analogous properties hold for the symmetric monoidal ∞-category Spc_* of pointed spaces with smash product and monoidal unit the 0-sphere S^0, and for the symmetric monoidal ∞-category Sp of spectra with smash product and monoidal unit the sphere spectrum $\mathbb{S}$, if one assumes moreover that $\mathscr{C}$ is respectively pointed or stable (see [HA, Proposition 4.8.2.11 and Corollary 4.8.2.19]).

Particular examples to keep in mind of such symmetric monoidal functors as described abstractly above are the functor $(-)_+ : \mathrm{Spc} \to \mathrm{Spc}_*$ adding a disjoint basepoint, the suspension spectrum functor $\Sigma^\infty : \mathrm{Spc}_* \to \mathrm{Sp}$, and their composition $\Sigma^\infty_+ : \mathrm{Spc} \to \mathrm{Sp}$. We generally suppress these essentially unique functors in the notation; for instance, $\mathbb{E}_\infty$ denotes both the $\mathbb{E}_\infty$-operad in spaces and the $\mathbb{E}_\infty$-operad $\Sigma^\infty_+\mathbb{E}_\infty$ in spectra.

We are now ready to discuss some examples of operads in spaces. Our discussion is rather informal: rather than giving definitions, we describe important properties of these operads. We refer to [HA] for a formal account, albeit using a different model. Recall that all our operads are nonunital (Remark 2.4).

Example 2.8 The *commutative operad* Com has the one-point space $\mathrm{Com}(n) := *$ in all levels $n \geqslant 1$. The composition maps of Com are all equivalent to the identity of the one-point space (which is indeed the only endomorphism).

Example 2.9 The *associative operad* Assoc is defined by $\mathrm{Assoc}(n) := \Sigma_n$, the symmetric group on n letters, in each level $n \geqslant 1$. Equivalently, for a finite set I, $\mathrm{Assoc}(I)$ can be described as the set of linear orders on I. There is a natural Σ_n-action on $\mathrm{Assoc}(n)$, and composition is given by composition of permutations (see for instance [HM22, Example 1.5(b)] for a description of the combinatorics).

Example 2.10 For every $k \geqslant 1$, $\mathbb{E}_k$ denotes the *little k-cubes operad*. The essential features of the $\mathbb{E}_k$-operads are already present in the case of topological operads (that is, in the 1-category of topological spaces), which we discuss here; the ∞-operad $\mathbb{E}_k$ is defined by Lurie in [HA, § 5.1]. Let $I^k = [0,1]^k$ denote the unit k-dimensional cube. A continuous map $I^k \to I^k$ is a *rectilinear embedding* if it is of the form

$$(x_1, \dots, x_k) \mapsto (t_1 x_1 + a_1, \dots, t_k x_k + a_k)$$

for some $t_i > 0$ and a_i. Specifying a rectilinear embedding $c : I^k \to I^k$ is equivalently specifying a 'little cube' $[a_1, b_1] \times \cdots \times [a_k, b_k] \subseteq [0,1]^k$. We define the space $\mathbb{E}_k(n)$ as the subspace of $\mathrm{Map}(\coprod_{i=1}^n I^k, I^k)$ of those maps that are of the form $c_1 \amalg \dots \amalg c_n$ such that each c_i is a rectilinear embedding and the images of the c_i on the interior $(0,1)^k$ of I^k do not overlap. The symmetric group Σ_n acts on $\mathbb{E}_k(n)$ by permutation of the 'little cubes' c_i. Composition in $\mathbb{E}_k$ is given by composition of functions.

Example 2.11 Any little k-cube $[a_1, b_1] \times \cdots \times [a_k, b_k] \subseteq [0,1]^k$ gives rise to a little $(k+1)$-cube by adding the entire interval $[a_{k+1}, b_{k+1}] := [0,1]$ in the next dimension. This procedure induces an inclusion of operads $\mathbb{E}_k \hookrightarrow \mathbb{E}_{k+1}$. The $\mathbb{E}_\infty$*-operad* is defined as the colimit of the sequence

$$\mathbb{E}_1 \hookrightarrow \mathbb{E}_2 \hookrightarrow \mathbb{E}_3 \hookrightarrow \dots .$$

It turns out that the spaces $\mathbb{E}_\infty(n)$ are contractible for all $n \geqslant 1$. Correspondingly, the $\mathbb{E}_\infty$-operad is homotopy equivalent to the commutative operad Com, which has the same property. Thus, in the ∞-category of operads in the ∞-category of spaces, there is an equivalence $\mathrm{Com} \simeq \mathbb{E}_\infty$. Similarly, one can see that the $\mathbb{E}_1$-operad is equivalent to the associative operad Assoc.

Notation 2.12 In light of the equivalences $\mathrm{Com} \simeq \mathbb{E}_\infty$ and $\mathrm{Assoc} \simeq \mathbb{E}_1$, we consistently write $\mathbb{E}_\infty$ for the commutative operad and $\mathbb{E}_1$ for the associative operad in the ∞-categorical setting.

3 Stable operadic suspension

In this section, we discuss operadic suspension in symmetric monoidal ∞-categories $\mathscr{C}$ that are *stable*, and in particular in the category Sp of spectra. We first define the *operadic suspension* of the $\mathbb{E}_\infty$-operad in $\mathscr{C}$ as the endomorphism operad $\mathbf{s}\mathbb{E}_\infty := \mathrm{End}(\mathbb{S}^{-1})$ of the desuspended monoidal unit $\mathbb{S}$. For a general operad $\mathbf{O}$ in $\mathscr{C}$, we simply define the operadic suspension of $\mathbf{O}$ to be the levelwise tensor product $\mathbf{sO} := \mathbf{O} \otimes_{\mathrm{lev}} \mathbf{s}\mathbb{E}_\infty$. The associated monads of $\mathbf{O}$ and its suspension $\mathbf{sO}$ satisfy

$$\mathrm{free}_{\mathbf{sO}} \simeq \Sigma^{-1} \mathrm{free}_{\mathbf{O}} \Sigma,$$

and it follows that there is an equivalence between $\mathbf{sO}$-algebra structures on an object $X \in \mathscr{C}$ and $\mathbf{O}$-algebra structures on its suspension ΣX (cf. [HL24, § 3.1]).

Our goal is to characterise operadic suspension by a universal property. We discuss two potential approaches to such a characterisation.

First, we discuss the approach via a *suspension morphism* $\sigma : \mathbf{O} \to \mathbf{sO}$. At the level of monads, there is a natural construction of a map $T \to \Sigma^{-1}T\Sigma$, providing in particular a map

$$\mathrm{free}_{\mathbf{O}} \to \Sigma^{-1}\mathrm{free}_{\mathbf{O}}\Sigma \simeq \mathrm{free}_{\mathbf{sO}}$$

between the monads associated to $\mathbf{O}$ and its suspension $\mathbf{sO}$. We investigate a conjecture of Heuts–Land [HL24, § 3.5], saying that this map of monads should arise from a map $\mathbf{O} \to \mathbf{sO}$ of operads. We show that the positive answer to this question of *ibid.* in the case of the $\mathbb{E}_n$-operads extends to a positive result for the $\mathbb{E}_\infty$-operad. The proof boils down to showing that all constructions involved commute with sifted colimits, and applying this to the sifted colimit $\mathbb{E}_\infty \simeq \mathrm{colim}_n \mathbb{E}_n$. We discuss two additional properties one could demand the suspension morphism to satisfy, and show that imposing these determine suspension up to equivalence.

Second, we observe that the operad $\mathbf{s}\mathbb{E}_\infty$ in spectra is invertible with respect to the levelwise tensor product: it follows from the equivalence $\mathbb{S}^1 \otimes \mathbb{S}^{-1} \simeq \mathbb{S}$ that the 'desuspended $\mathbb{E}_\infty$-operad' $\mathbf{s}^{-1}\mathbb{E}_\infty := \mathrm{End}(\mathbb{S}^1)$ is inverse to $\mathbf{s}\mathbb{E}_\infty$, in the sense that there is an equivalence

$$\mathbf{s}\mathbb{E}_\infty \otimes_{\mathrm{lev}} \mathbf{s}^{-1}\mathbb{E}_\infty \simeq \mathbb{E}_\infty$$

to the monoidal unit. Writing $\mathbf{s}^d\mathbb{E}_\infty$ for the d-fold suspension (or $-d$-fold desuspension if d is negative) of $\mathbb{E}_\infty$, we conjecture that every nonunital operad in spectra that is invertible with respect to the levelwise tensor product is equivalent to $\mathbf{s}^d\mathbb{E}_\infty$ for some $d \in \mathbb{Z}$. Further, we discuss a weaker condition that we call 'quasi-invertibility' and that is easier to check in practice; we investigate the relation between invertibility and quasi-invertibility.

Remark 3.1 The notion of operadic suspension has already been studied in the context of Koszul duality in the linear case, in the category of chain complexes over an ordinary commutative ring R, for instance by Berger–Fresse [BF04] and by Fresse [Fre11], using point-set models; see also [LV12, § 7.2.2]. One similarly defines operadic suspension in this case by tensoring with the endomorphism operad $\mathrm{End}(R[-1])$ of the desuspension of the monoidal unit $R[0]$ (however, the cited sources flip the suspension sign and call this *desuspension*). In the epilogue of [Fre11], Fresse shows that the suspension morphism in the R-linear case agrees with the action of $\mathbb{E}_\infty$ on reduced spherical cochains of S^1, as is the case for the suspension morphism we discuss in Definition 3.3 (see [ABH25]).

3.1 Operadic suspension

We start with a definition of operadic suspension in the stable case.

Let $\mathscr{C}$ be a presentable, closed symmetric monoidal ∞-category. Moreover, we assume $\mathscr{C}$ to be *stable* in this section and we denote the monoidal unit by $\mathbb{S}$; for the n-fold suspension of $\mathbb{S}$ (where $n \in \mathbb{Z}$), we write $\mathbb{S}^n := \Sigma^n\mathbb{S}$. The right adjoint of $X \otimes - : \mathscr{C} \to \mathscr{C}$ is denoted $\mathrm{map}(X, -)$ for every $X \in \mathscr{C}$.

Definition 3.2 The *operadic suspension* of the $\mathbb{E}_\infty$-operad in $\mathscr{C}$ is the endomorphism operad

$$\mathbf{s}\mathbb{E}_\infty := \mathrm{End}(\mathbb{S}^{-1})$$

of the desuspension of $\mathbb{S}$. The *operadic suspension functor* is defined by

$$\mathbf{s} := \mathbf{s}\mathbb{E}_\infty \otimes_{\mathrm{lev}} - : \mathrm{Opd}(\mathscr{C}) \to \mathrm{Opd}(\mathscr{C}).$$

In particular, the operadic suspension of an operad $\mathbf{O}$ in $\mathscr{C}$ is $\mathbf{sO} = \mathbf{s}\mathbb{E}_\infty \otimes_{\mathrm{lev}} \mathbf{O}$.

Explicitly, the underlying symmetric sequence of $\mathbf{s}\mathbb{E}_\infty$ is given by

$$\mathbf{s}\mathbb{E}_\infty(n) = \mathrm{map}(\mathbb{S}^{-n}, \mathbb{S}^{-1}) \simeq \mathbb{S}^{n-1} \simeq \Sigma^{-1}(\mathbb{S}^1)^{\otimes n}$$

where Σ_n acts by permutation of the $\mathbb{S}^1$ factors. Alternatively, $\mathbf{s}\mathbb{E}_\infty(n)$ can be described as the representation sphere $\mathbb{S}^{\rho_n}$, the suspension spectrum of S^{ρ_n} which we define in § 4.1; see also [HL24, Remark 3.2].

The monad associated to the suspended operad $\mathbf{sO}$ satisfies

$$\mathrm{free}_{\mathbf{sO}} \simeq \Sigma^{-1}\mathrm{free}_{\mathbf{O}}\Sigma,$$

and thus there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Alg}_{\mathbf{sO}}(\mathscr{C}) & \xrightarrow{\simeq} & \mathrm{Alg}_{\mathbf{O}}(\mathscr{C}) \\ {\scriptstyle\mathrm{fgt}}\downarrow & & \downarrow{\scriptstyle\mathrm{fgt}} \\ \mathscr{C} & \xrightarrow[\Sigma]{\simeq} & \mathscr{C} \end{array}$$

providing an equivalence between $\mathbf{sO}$-algebra structures on X and $\mathbf{O}$-algebra structures on ΣX.

Note that the object $\mathbb{S}^{-1}$ is invertible: tensoring it with $\mathbb{S}^1$ results in the monoidal unit $\mathbb{S}$. It follows that the operad $\mathbf{s}\mathbb{E}_\infty$ is invertible with respect to the levelwise tensor product, and its inverse is $\mathbf{s}^{-1}\mathbb{E}_\infty := \mathrm{End}(\mathbb{S}^1)$. (We will pursue this property further in § 3.3.) Tensoring with $\mathbf{s}^{-1}\mathbb{E}_\infty$ provides an *operadic desuspension* $\mathbf{s}^{-1}$ on $\mathrm{Opd}(\mathscr{C})$, which is inverse to suspension. The functor of (1) then gives an equivalence

$$\mathbb{S}^{-1} \otimes - : \mathrm{Alg}_{\mathbf{O}}(\mathscr{C}) \xrightarrow{\simeq} \mathrm{Alg}_{\mathbf{sO}}(\mathscr{C}), \tag{2}$$

where $\mathbb{S}^{-1}$ is equipped with the canonical $\mathbf{s}\mathbb{E}_\infty$-algebra structure (given by the identity $\mathbf{s}\mathbb{E}_\infty \to \mathrm{End}(\mathbb{S}^{-1})$), with inverse given by tensoring with $\mathbb{S}^1 \in \mathrm{Alg}_{\mathbf{s}^{-1}\mathbb{E}_\infty}(\mathscr{C})$.

3.2 The suspension morphism

In this section, we discuss one approach to characterising operadic suspension in the stable case: via the suspension morphism, as suggested by Heuts–Land [HL24, § 3.5]. The idea is to focus on the equivalence of monads $\mathrm{free}_{\mathbf{sO}} \simeq \Sigma^{-1}\mathrm{free}_{\mathbf{O}}\Sigma$ associated to an operad $\mathbf{O}$. For a general (reduced) monad T on a stable category $\mathscr{C}$, conjugating with the suspension functor on $\mathscr{C}$ gives a monad $\Sigma^{-1}T\Sigma$ on $\mathscr{C}$, and there is a natural *suspension morphism* $T \to \Sigma^{-1}T\Sigma$ between these monads. Specialising to the case that T is the monad $\mathrm{free}_{\mathbf{O}}$ associated to an operad, one thus obtains a map

$$\mathrm{free}_{\mathbf{O}} \to \Sigma^{-1}\mathrm{free}_{\mathbf{O}}\Sigma \simeq \mathrm{free}_{\mathbf{sO}}.$$

Although the functor $\mathrm{free} : \mathrm{Opd}(\mathscr{C}) \to \mathrm{Mnd}(\mathscr{C})$ is not fully faithful in general, the map of monads $\mathrm{free}_{\mathbf{O}} \to \mathrm{free}_{\mathbf{sO}}$ should come from a map of operads $\mathbf{O} \to \mathbf{sO}$. Indeed, there is a natural construction of a map $\mathbb{E}_\infty \to \mathbf{s}\mathbb{E}_\infty$, and tensoring with this map defines a *suspension morphism* $\mathbf{O} \to \mathbf{sO}$ at the level of operads (this is [ABH25, Definition 2.4]). One might expect that this suspension morphism of operads should induce the corresponding suspension morphism of monads.

In the specific case of the $\mathbb{E}_n$-operads (for finite n), Heuts–Land construct a suspension morphism $\mathbb{E}_n \to \mathbf{s}\mathbb{E}_n$ with this property; it is shown by Antolín-Camarena–Brantner–Heuts [ABH25] that this construction is equivalent to theirs, providing a positive answer for the $\mathbb{E}_n$-operads to the question as phrased above.

We show that all constructions involved preserve sifted colimits to conclude that also the $\mathbb{E}_\infty$-operad satisfies this property. The $\mathbb{E}_\infty$-operad, being the monoidal unit with respect to the levelwise tensor product, should be in some way fundamental in this story. The question is still open, however, whether one can extend the positive result for the $\mathbb{E}_\infty$-operad to all operads (see Remark 3.7).

The question is also still open to what extent this property characterises operadic suspension. We conclude the section by showing that a natural suspension morphism satisfying this property for the $\mathbb{E}_n$-operads and some additional properties is essentially unique. Informally, these properties are (1) that the suspension morphism is natural and determined by what it does to the $\mathbb{E}_\infty$-operad; (2) that the underlying symmetric sequence of $\mathbf{s}\mathbb{E}_\infty$ consists of representation spheres $\mathbb{S}^{\rho_n}$.

Suspension morphism for monads Let T be a reduced monad on $\mathscr{C}$, that is, satisfying $T(0) \simeq 0$. Then the underlying object of the free T-algebra on the zero object of $\mathscr{C}$ is $\mathrm{fgt}_T\mathrm{free}_T(0) = T(0) \simeq 0$, and $\mathrm{free}_T(0)$ is the zero object of the ∞-category $\mathrm{Alg}_T(\mathscr{C})$ of T-algebras. The loop functor Ω_T on $\mathrm{Alg}_T(\mathscr{C})$, defined on a T-algebra X as the pullback of $\mathrm{free}_T(0) \to X$ along itself, then satisfies

$$\mathrm{fgt}_T\Omega_T \simeq \Omega\mathrm{fgt}_T : \mathrm{Alg}_T(\mathscr{C}) \to \mathscr{C},$$

since the right adjoint fgt_T preserves the defining pullback square. Hence, we have a loops–suspension adjunction on $\mathrm{Alg}_T(\mathscr{C})$ which commutes with the loops–suspension adjunction of $\mathscr{C}$ as in the diagrams

$$\begin{array}{ccc} \mathrm{Alg}_T(\mathscr{C}) & \xrightarrow{\Sigma_T} & \mathrm{Alg}_T(\mathscr{C}) \\ {\scriptstyle \mathrm{free}_T}\uparrow & & \uparrow{\scriptstyle \mathrm{free}_T} \\ \mathscr{C} & \xrightarrow[\Sigma]{} & \mathscr{C} \end{array} \qquad \begin{array}{ccc} \mathrm{Alg}_T(\mathscr{C}) & \xleftarrow{\Omega_T} & \mathrm{Alg}_T(\mathscr{C}) \\ {\scriptstyle \mathrm{fgt}_T}\downarrow & & \downarrow{\scriptstyle \mathrm{fgt}_T} \\ \mathscr{C} & \xleftarrow[\Omega]{} & \mathscr{C} \end{array}$$

of respectively left and right adjoints. From the unit of the adjunction $\Sigma_T \dashv \Omega_T$, we obtain a map

$$\sigma_T : T \to \mathrm{fgt}_T\Omega_T\Sigma_T\mathrm{free}_T \simeq \Omega\mathrm{fgt}_T\mathrm{free}_T\Sigma = \Omega T\Sigma,$$

which we call the *suspension morphism* of the reduced monad T. This construction is natural in the monad T, or in other words, it defines a functor

$$\sigma : \mathrm{Mnd}^{\mathrm{red}}(\mathscr{C}) \to \mathrm{Fun}(\Delta^1, \mathrm{Mnd}^{\mathrm{red}}(\mathscr{C}))$$

from the ∞-category of reduced monads on $\mathscr{C}$ to its arrow category. For the reduced monad $\mathrm{free}_{\mathbf{O}}$ of a nonunital operad $\mathbf{O}$, this construction provides a map of monads

$$\sigma_{\mathrm{free}_{\mathbf{O}}} : \mathrm{free}_{\mathbf{O}} \to \Omega\mathrm{free}_{\mathbf{O}}\Sigma \simeq \mathrm{free}_{\mathbf{sO}}\,.$$

Suspension morphism for operads Heuts–Land [HL24] conjecture that the map $\sigma_{\mathrm{free}_{\mathbf{O}}}$ arises from a map of operads $\mathbf{O} \to \mathbf{sO}$, thus a *suspension morphism* of the operad $\mathbf{O}$. Note that the functor free : $\mathrm{SSeq}(\mathscr{C}) \to \mathrm{End}(\mathscr{C})$ is in general not fully faithful; a counterexample in the case $\mathscr{C} = \mathrm{Sp}$ is given in [HL24, § 3.2]. Heuts–Land do show in [HL24, Theorem 3.8], however, that the functor $A \mapsto \mathrm{free}_A$ is fully faithful on a full monoidal subcategory of symmetric sequences in spectra, those with 'nilpotent Euler classes'. Among the operads with nilpotent Euler classes are in particular the $\mathbb{E}_n$-operads (but not $\mathbb{E}_\infty$) and their operadic suspensions. Hence, the map $\sigma_{\mathrm{free}_{\mathbb{E}_n}} : \mathrm{free}_{\mathbb{E}_n} \to \mathrm{free}_{\mathbf{s}\mathbb{E}_n}$ must indeed come from a map of operads $\mathbb{E}_n \to \mathbf{s}\mathbb{E}_n$. A specific construction of such a map σ with the property that free_σ is equivalent to $\sigma_{\mathrm{free}_{\mathbb{E}_n}}$ is given for the $\mathbb{E}_n$-operads in spectra in [HL24, § 3.4].

Antolín-Camerena–Brantner–Heuts [ABH25] give a more general construction of a suspension morphism $\sigma_{\mathbf{O}} : \mathbf{O} \to \mathbf{sO}$ for all nonunital operads $\mathbf{O}$ in spectra. The idea is to first construct the suspension morphism $\sigma_{\mathbb{E}_\infty}$ for the $\mathbb{E}_\infty$-operad, which plays a central role in Definition 3.2 of operadic suspension; then, tensoring $\sigma_{\mathbb{E}_\infty}$ levelwise with the identity of $\mathbf{O}$ gives a suspension morphism $\sigma_{\mathbf{O}}$.

Definition 3.3 Consider the canonical $\mathbb{E}_\infty$-algebra structure on the monoidal unit $\mathbb{S}$ of $\mathscr{C}$. Applying the loop functor $\Omega_{\mathbb{E}_\infty}$ internal to the ∞-category of $\mathbb{E}_\infty$-algebras in $\mathscr{C}$, we obtain an $\mathbb{E}_\infty$-algebra in $\mathscr{C}$ with underlying object $\Omega\mathbb{S} = \mathbb{S}^{-1}$. The *suspension morphism* of $\mathbb{E}_\infty$ is the corresponding map of operads

$$\sigma_{\mathbb{E}_\infty} : \mathbb{E}_\infty \to \mathrm{End}(\mathbb{S}^{-1}) = \mathbf{s}\mathbb{E}_\infty.$$

For any nonunital operad $\mathbf{O}$ in $\mathscr{C}$, the *suspension morphism* of $\mathbf{O}$ is the map

$$\sigma_{\mathbf{O}} : \mathbf{O} \simeq \mathbf{O} \otimes_{\mathrm{lev}} \mathbb{E}_\infty \xrightarrow{\mathrm{id}_{\mathbf{O}} \otimes_{\mathrm{lev}} \sigma_{\mathbb{E}_\infty}} \mathbf{O} \otimes_{\mathrm{lev}} \mathbf{s}\mathbb{E}_\infty \simeq \mathbf{sO}.$$

Antolín-Camerena–Brantner–Heuts show in [ABH25, Proposition 2.5] that the composite

$$\mathrm{Alg}_{\mathbf{O}}(\mathrm{Sp}) \xrightarrow{\mathbb{S}^{-1} \otimes -} \mathrm{Alg}_{\mathbf{sO}}(\mathrm{Sp}) \xrightarrow{\sigma_{\mathbf{O}}^*} \mathrm{Alg}_{\mathbf{O}}(\mathrm{Sp})$$

of the equivalence (2) and restriction along the suspension morphism of $\mathbf{O}$ is equivalent to the loop functor $\Omega_{\mathbf{O}}$ on $\mathrm{Alg}_{\mathbf{O}}(\mathrm{Sp})$. Since Heuts–Land show that the suspension morphisms they constructed for the $\mathbb{E}_n$-operads in spectra satisfy the same property, these two definitions are equivalent.

Characterising suspension Given the multiple constructions of an operadic suspension morphism $\mathbf{O} \to \mathbf{sO}$ in the literature (see also § 4 for a discussion of different constructions in the unstable setting), we would like a universal property that characterises operadic suspension up to equivalence. In the stable case, Heuts–Land expect the suspension morphism $\sigma : \mathbf{O} \to \mathbf{sO}$ of a nonunital operad $\mathbf{O}$ in $\mathscr{C}$ to be characterised by the following property [HL24, property (F)]:

(*) There is a canonical equivalence $\Sigma\mathrm{free}_{\mathbf{sO}} \simeq \mathrm{free}_{\mathbf{O}}\Sigma$ in $\mathrm{End}(\mathscr{C})$. The adjoint map $\mathrm{free}_{\mathbf{sO}} \to \Sigma^{-1}\mathrm{free}_{\mathbf{O}}\Sigma$ (which is also an equivalence) refines to a map of monads making the diagram of monads

$$\begin{array}{ccc} & \mathrm{free}_{\mathbf{O}} & \\ {\scriptstyle \mathrm{free}_{\sigma}} \swarrow & & \searrow {\scriptstyle \sigma_{\mathrm{free}_{\mathbf{O}}}} \\ \mathrm{free}_{\mathbf{sO}} & \xrightarrow[\simeq]{} & \Sigma^{-1}\mathrm{free}_{\mathbf{O}}\Sigma \end{array}$$

commute.

In particular, the maps of monads

$$\mathrm{free}_{\sigma} : \mathrm{free}_{\mathbf{O}} \to \mathrm{free}_{\mathbf{sO}} \qquad \text{and} \qquad \sigma_{\mathrm{free}_{\mathbf{O}}} : \mathrm{free}_{\mathbf{O}} \to \Sigma^{-1}\mathrm{free}_{\mathbf{O}}\Sigma$$

should be equivalent in the arrow category of monads on $\mathscr{C}$.

Heuts–Land show in [HL24, Theorem 3.11] that property $\circledast$ is satisfied by their construction of the suspension morphism for the spectral $\mathbb{E}_n$-operads, so by [ABH25, Remark 2.6], it is also satisfied by the suspension morphisms $\sigma_{\mathbb{E}_n}$ of Definition 3.3.

Theorem 3.4 ([HL24]) *The suspension morphism of the nonunital $\mathbb{E}_n$-operad in spectra satisfies property $\circledast$ for all $n \geqslant 1$.*

Using the filtration

$$\mathbb{E}_1 \hookrightarrow \mathbb{E}_2 \hookrightarrow \mathbb{E}_3 \hookrightarrow \ldots \to \mathbb{E}_\infty$$

of the $\mathbb{E}_\infty$-operad by the $\mathbb{E}_n$-operads, we can extend this result to the $\mathbb{E}_\infty$-operad.

Proposition 3.5 *The suspension morphism of the nonunital $\mathbb{E}_\infty$-operad in spectra satisfies property $\circledast$.*

Proof We will use the filtration of $\mathbb{E}_\infty$ by the $\mathbb{E}_n$-operads and the fact that the $\mathbb{E}_n$-operads satisfy property $\circledast$: it suffices to prove that the functors $\mathbf{O} \mapsto \mathrm{free}_{\sigma_{\mathbf{O}}}$ and $\mathbf{O} \mapsto \sigma_{\mathrm{free}_{\mathbf{O}}}$ preserve sifted colimits of nonunital operads. The equivalence $\sigma_{\mathrm{free}_{\mathbb{E}_\infty}} \simeq \mathrm{free}_{\sigma_{\mathbb{E}_\infty}}$ then follows from the corresponding equivalence for the $\mathbb{E}_n$-operads, because both functors preserve the colimit $\mathbb{E}_\infty \simeq \mathrm{colim}_n \, \mathbb{E}_n$.

The former functor $\mathbf{O} \mapsto \mathrm{free}_{\sigma_{\mathbf{O}}}$ preserves sifted colimits since the suspension morphism $\sigma_{\mathbf{O}} \simeq \mathrm{id}_{\mathbf{O}} \otimes \sigma_{\mathbb{E}_\infty}$ is natural in $\mathbf{O}$ and the tensor product is assumed to preserve colimits in both variables, and because $\mathrm{free} : \mathrm{Opd}(\mathscr{C}) \to \mathrm{Mnd}(\mathscr{C})$ preserves sifted colimits by Lemma 2.6.

The latter functor $\mathbf{O} \mapsto \sigma_{\mathrm{free}_{\mathbf{O}}}$ is the composition of the functors

$$\mathrm{Opd}^{\mathrm{nu}}(\mathscr{C}) \xrightarrow{\mathrm{free}} \mathrm{Mnd}^{\mathrm{red}}(\mathscr{C}) \xrightarrow{\sigma} \mathrm{Fun}(\Delta^1, \mathrm{Mnd}^{\mathrm{red}}(\mathscr{C}))$$

of which the former preserves sifted colimits by Lemma 2.6. Since the monads $\mathrm{free}_{\mathbf{O}}$ preserve sifted colimits themselves as well, we can further factor the composite through the ∞-category $\mathrm{Mnd}^{\Sigma,\mathrm{red}}(\mathscr{C})$ of the monads on $\mathscr{C}$ that are reduced and preserve sifted colimits. To see that σ preserves sifted colimits when restricted to this subcategory, it suffices to check this after evaluating at the endpoints of Δ^1. At the endpoint 0, we recover the identity functor, which certainly preserves sifted colimits. At the other endpoint 1, we have the functor $T \mapsto \Omega T \Sigma$; to show that this functor preserves sifted colimits, we use an argument similar to the one in Lemma 2.6: we have a commutative square

$$\begin{array}{ccc} \mathrm{Mnd}^{\Sigma,\mathrm{red}}(\mathscr{C}) & \xrightarrow{\Omega - \Sigma} & \mathrm{Mnd}^{\Sigma,\mathrm{red}}(\mathscr{C}) \\ {\scriptstyle \mathrm{fgt}} \downarrow & & \downarrow {\scriptstyle \mathrm{fgt}} \\ \mathrm{End}^{\Sigma,\mathrm{red}}(\mathscr{C}) & \xrightarrow[\Omega - \Sigma]{} & \mathrm{End}^{\Sigma,\mathrm{red}}(\mathscr{C}) \end{array}$$

where the vertical forgetful functors create sifted colimits by [HA, Proposition 3.2.3.1] and the bottom horizontal functor preserves all colimits (since Σ and Ω are equivalences). $\square$

Remark 3.6 The proof of Proposition 3.5 shows more generally that for a sifted diagram $\mathbf{O} : I \to \mathrm{Opd}(\mathscr{C})$ of nonunital operads $\mathbf{O}_i$ satisfying property ⊛ for all $i \in I$ in a natural way (i.e., such that the equivalences $\mathrm{free}_{\sigma_{\mathbf{O}_i}} \simeq \sigma_{\mathrm{free}_{\mathbf{O}_i}}$ are natural in $i \in I$), the colimit of the diagram $\mathbf{O}$ again satisfies property ⊛.

Remark 3.7 In Proposition 3.5, we have proved that the $\mathbb{E}_\infty$-operad in spectra satisfies property ⊛. An open question remains if this result can be extended to all nonunital operads $\mathbf{O}$ in spectra. Since $\mathbb{E}_\infty$ is the monoidal unit for the levelwise tensor product and we defined suspension of a general operad by tensoring with the suspension of $\mathbb{E}_\infty$, we might suspect that, if property ⊛ holds for all operads, one can use the result for $\mathbb{E}_\infty$ to prove this. To that end, one could try to relate the monad $\mathrm{free}_{\mathbf{O}\otimes_{\mathrm{lev}}\mathbf{P}}$ associated to a levelwise tensor product $\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P}$ (specifically for $\mathbf{P} = \mathbb{E}_\infty$) to the monads $\mathrm{free}_{\mathbf{O}}$ and $\mathrm{free}_{\mathbf{P}}$ associated to $\mathbf{O}$ and $\mathbf{P}$; a potential relation does not immediately present itself, however.

Remark 3.8 Another open question is to what extent property ⊛ characterises operadic suspension. Given two suspension morphisms $\sigma : \mathbf{O} \to \mathbf{sO}$ and $\sigma' : \mathbf{O} \to \mathbf{s'O}$ satisfying property ⊛, one would directly obtain an equivalence $\mathrm{free}_\sigma \simeq \mathrm{free}_{\sigma'}$, but an equivalence $\sigma \simeq \sigma'$ does not immediately follow since $\mathrm{free} : \mathrm{Opd}(\mathscr{C}) \to \mathrm{Mnd}(\mathscr{C})$ is not fully faithful in general. It does follow for the $\mathbb{E}_n$-operads if $\mathbf{s}\mathbb{E}_n$ and $\mathbf{s}'\mathbb{E}_n$ have nilpotent Euler classes, as discussed below.

Although it is not clear whether suspension morphisms with property ⊛ are essentially unique, we can impose some additional (reasonable) conditions that do imply that any two suspension morphisms satisfying them are equivalent. We propose here the following two conditions (restricting to the case where $\mathscr{C}$ is the category of spectra), both clearly satisfied by the suspension morphism of Definition 3.3:

① There is a suspension $\sigma_{\mathbb{E}_\infty} : \mathbb{E}_\infty \to \mathbf{s}\mathbb{E}_\infty$ for the $\mathbb{E}_\infty$-operad (the monoidal unit), and the suspension morphism is a natural transformation $\sigma : \mathrm{id} \to \mathbf{s}$ to an endofunctor $\mathbf{s}$ on the category of operads such that $\sigma \simeq \mathrm{id} \otimes_{\mathrm{lev}} \sigma_{\mathbb{E}_\infty}$ (via the canonical equivalence $\mathrm{id} \simeq \mathrm{id} \otimes_{\mathrm{lev}} \mathbb{E}_\infty$).

② The underlying symmetric sequence of the suspension $\mathbf{s}\mathbb{E}_\infty$ of the $\mathbb{E}_\infty$-operad is equivalent to that of $\mathrm{End}(\mathbb{S}^{-1})$.

Before showing that these assumptions determine an essentially unique suspension morphism, let us justify them. The first implies that operadic suspension is functorial and the suspension morphism is natural. It follows from the equivalence $\sigma \simeq \mathrm{id} \otimes_{\mathrm{lev}} \sigma_{\mathbb{E}_\infty}$ that there is a natural equivalence

$$\sigma_{\mathbf{O}} \otimes_{\mathrm{lev}} \mathrm{id}_{\mathbf{P}} \simeq \sigma_{\mathbf{O}\otimes_{\mathrm{lev}}\mathbf{P}} \simeq \mathrm{id}_{\mathbf{O}} \otimes_{\mathrm{lev}} \sigma_{\mathbf{P}},$$

and in particular a natural equivalence

$$\mathbf{sO} \otimes_{\mathrm{lev}} \mathbf{P} \simeq \mathbf{s}(\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P}) \simeq \mathbf{O} \otimes_{\mathrm{lev}} \mathbf{sP}$$

for the codomains, analogous to the natural equivalence $\Sigma X \otimes Y \simeq \Sigma(X \otimes Y) \simeq X \otimes \Sigma Y$ for suspension of spectra (or for pointed spaces). Like suspension of pointed spaces, spectra, or chain complexes is defined by tensoring with the right notion of a 'sphere', property ① essentially says that $\mathbf{s}\mathbb{E}_\infty$ plays the role of a sphere operad (as, indeed, the unstable operads discussed in § 4 are called). Operadic suspension and the suspension morphism are thus entirely determined by their value on the monoidal unit $\mathbb{E}_\infty$.

The second condition is obviously necessary for any construction of operadic suspension in order to agree with that of Definition 3.2. However, it is (*a priori*) definitely weaker than asking for an equivalence of operads $\mathbf{s}\mathbb{E}_\infty \simeq \mathrm{End}(\mathbb{S}^{-1})$. Property ② is equivalent to demanding the terms $\mathbf{s}\mathbb{E}_\infty(n)$ be Σ_n-equivariantly equivalent to the representation spheres $\mathbb{S}^{\rho_n}$. The utility of this property lies in the following consequence, phrased in terms of 'nilpotent Euler classes' in the sense of [HL24, Definition 3.4]:

②′ A symmetric sequence A has nilpotent Euler classes if and only if its suspension $\mathbf{s}A = A \otimes_{\mathrm{lev}} \mathbf{s}\mathbb{E}_\infty$ has nilpotent Euler classes.

Indeed, this follows from the proof of [HL24, Lemma 3.6], which shows that the property of a symmetric sequence of having nilpotent Euler classes is closed under tensoring with the symmetric sequence formed by the representation spheres $\mathbb{S}^{\rho_n}$. Property ②′ in turn has the following useful consequence, as follows from [HL24, Theorem 3.8]:

②″ The functor free : $\mathrm{Opd}(\mathrm{Sp}) \to \mathrm{Mnd}(\mathrm{Sp})$ is fully faithful on the full subcategory spanned by the $\mathbb{E}_n$-operads and their suspensions $\mathbf{s}\mathbb{E}_n$.

In particular, an equivalence $\mathrm{free}_{\sigma_{\mathbb{E}_n}} \simeq \mathrm{free}_{\sigma'_{\mathbb{E}_n}}$ of maps of monads would imply an equivalence $\sigma_{\mathbb{E}_n} \simeq \sigma'_{\mathbb{E}_n}$ of suspension morphisms if σ and σ' satisfy property ②″. In practice, it will probably be easier to check property ② for a given construction of operadic suspension, which is a property of the symmetric sequences of the suspensions $\mathbf{s}\mathbb{E}_n$.

We conclude by showing that additionally assuming properties ① and ② ensure essential uniqueness of the suspension morphism. Given these properties, we only need to assume property ⊛ holds for the $\mathbb{E}_n$-operads.

Proposition 3.9 *There is an essentially unique natural endofunctor* $\mathbf{s}$ *on the* ∞*-category* $\mathrm{Opd}(\mathrm{Sp})$ *of nonunital operads in spectra, together with a natural transformation* $\sigma : \mathrm{id} \to \mathbf{s}$ *satisfying properties ① and ②, such that* $\sigma_{\mathbb{E}_n} : \mathbb{E}_n \to \mathbf{s}\mathbb{E}_n$ *satisfies property ⊛ for all* $n \geqslant 1$.

Proof Existence is provided by Definition 3.3.

For uniqueness, let $\sigma : \mathrm{id} \to \mathbf{s}$ and $\sigma' : \mathrm{id} \to \mathbf{s}'$ be any two such natural operadic suspension morphisms. By property ⊛, we obtain equivalences

$$\mathrm{free}_{\sigma_{\mathbb{E}_n}} \simeq \sigma_{\mathrm{free}_{\mathbb{E}_n}} \simeq \mathrm{free}_{\sigma'_{\mathbb{E}_n}}$$

(where the middle map is the suspension morphism of the monad $\mathrm{free}_{\mathbb{E}_n}$). Invoking property ② (or rather, its consequence ②″) then gives us equivalences $\sigma_{\mathbb{E}_n} \simeq \sigma'_{\mathbb{E}_n}$. Property ① tells us that σ and σ' are given by tensoring with respectively $\sigma_{\mathbb{E}_\infty}$ and $\sigma'_{\mathbb{E}_\infty}$, so it follows that they preserve colimits since the tensor product preserves them separately in each variable. In particular, for the colimit $\mathbb{E}_\infty \simeq \mathrm{colim}_n\, \mathbb{E}_n$, we get an equivalence

$$\sigma_{\mathbb{E}_\infty} \simeq \sigma_{\mathrm{colim}_n \mathbb{E}_n} \simeq \operatorname*{colim}_n \sigma_{\mathbb{E}_n} \simeq \operatorname*{colim}_n \sigma'_{\mathbb{E}_n} \simeq \sigma'_{\mathrm{colim}_n \mathbb{E}_n} \simeq \sigma'_{\mathbb{E}_\infty}.$$

For any nonunital operad $\mathbf{O}$, we now have a natural equivalence

$$\sigma_{\mathbf{O}} \simeq \mathrm{id}_{\mathbf{O}} \otimes_{\mathrm{lev}} \sigma_{\mathbb{E}_\infty} \simeq \mathrm{id}_{\mathbf{O}} \otimes_{\mathrm{lev}} \sigma'_{\mathbb{E}_\infty} \simeq \sigma'_{\mathbf{O}},$$

finishing the proof. □

Remark 3.10 In light of property ①, the focus should be on the suspension morphism $\sigma_{\mathbb{E}_\infty} : \mathbb{E}_\infty \to \mathbf{s}\mathbb{E}_\infty$ of the monoidal unit $\mathbb{E}_\infty$, since the suspension morphism for other operads is simply given by tensoring the identity with $\sigma_{\mathbb{E}_\infty}$. We can then rephrase Proposition 3.9 as follows: there is an essentially unique morphism $\sigma_{\mathbb{E}_\infty} : \mathbb{E}_\infty \to \mathbf{s}\mathbb{E}_\infty$ out of $\mathbb{E}_\infty$ such that there are equivalences $\mathbf{s}\mathbb{E}_\infty(n) \simeq \mathbb{S}^{\rho_n}$ in $\mathrm{Fun}(B\Sigma_n, \mathrm{Sp})$ and the morphisms $\mathrm{id}_{\mathbb{E}_n} \otimes_{\mathrm{lev}} \sigma_{\mathbb{E}_\infty}$ satisfy property ⊛.

3.3 Invertible operads

In this section, we discuss a second approach to characterising operadic suspension in the stable case. The starting point is the fact, observed in § 3.1, that the suspension $\mathbf{s}\mathbb{E}_\infty$ of the $\mathbb{E}_\infty$-operad is invertible with respect to the levelwise tensor product: there is an equivalence

$$\mathbf{s}\mathbb{E}_\infty \otimes_{\mathrm{lev}} \mathbf{s}^{-1}\mathbb{E}_\infty \simeq \mathbb{E}_\infty.$$

This property is analogous to the equivalence $\mathbb{S}^1 \otimes \mathbb{S}^{-1} \simeq \mathbb{S}$ of spectra, and in the category of spectra, the invertible objects are precisely the shifts $\mathbb{S}^d$ of the sphere spectrum for $d \in \mathbb{Z}$. Moreover, the set of equivalence classes of invertible spectra obtains a group structure from the tensor product, which is called the *Picard group* of spectra and is isomorphic to $\mathbb{Z}$; the suspended sphere spectrum $\mathbb{S}^1$ is then characterised up to equivalence by the fact that it generates the Picard group.

We conjecture that a similar statement holds for nonunital operads in spectra: the invertible operads are precisely the iterated suspensions $\mathbf{s}^d\mathbb{E}_\infty$ (for $d \in \mathbb{Z}$), and the Picard group of spectral operads is isomorphic to $\mathbb{Z}$ and generated by $\mathbf{s}\mathbb{E}_\infty$. This would provide a characterisation of operadic suspension up to equivalence.

In practice, showing an operad $\mathbf{O}$ is invertible requires specifying a large amount of data: one has to define another operad $\mathbf{P}$ and a map of operads $\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P} \to \mathbb{E}_\infty$ that is an equivalence. We study the relationship between invertibility and an *a priori* weaker condition that we call 'quasi-invertibility': an operad $\mathbf{O}$ is *quasi-invertible* if the terms $\mathbf{O}(n)$ are invertible and if the partial composition maps of $\mathbf{O}$ are equivalences. We show that the underlying symmetric sequence of a quasi-invertible operad in spectra is equivalent to that of $\mathbf{s}^d\mathbb{E}_\infty$ for some $d \in \mathbb{Z}$. Moreover, we show that any invertible operad in spectra is quasi-invertible, and we claim the converse should also be true. Using an alternative model of enriched ∞-operads that is used by Hoffbeck–Moerdijk [HM24], as certain presheaves on a category of trees, we discuss how to construct an inverse of a quasi-invertible operad; however, no comparison between enriched ∞-operads in terms of symmetric sequences (used throughout this thesis) and the model of *ibid.* is currently available.

The Picard group Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a symmetric monoidal ∞-category. An object $X \in \mathscr{C}$ is called *invertible* if there is an object $X^* \in \mathscr{C}$ such that $X \otimes X^* \simeq \mathbb{1}$. We write $\mathrm{Pic}(\mathscr{C})$ for the *Picard category* of $\mathscr{C}$, the full subcategory spanned by the invertible objects. Its ∞-groupoid core $\mathrm{pic}(\mathscr{C}) := \mathrm{Pic}(\mathscr{C})^{\simeq}$ is called the *Picard groupoid* or *Picard space* of $\mathscr{C}$, and the collection $\pi_0(\mathrm{pic}(\mathscr{C}))$ of isomorphism classes of invertible objects is the *Picard group* of $\mathscr{C}$. The group operation is given by the tensor product of $\mathscr{C}$, with unit $\mathbb{1}$. The Picard group is indeed a small group if $\mathscr{C}$ is presentable, cf. [MS16, Remark 2.1.4]. It is also abelian since the tensor product of $\mathscr{C}$ is assumed to be symmetric (more generally, if $\mathscr{C}$ is any monoidal ∞-category, the Picard group need not be abelian).

If $\mathscr{C}$ is stable and the tensor product commutes with colimits in both variables separately, there is a natural homomorphism

$$\mathbb{Z} \to \pi_0(\mathrm{pic}(\mathscr{C})), \quad n \mapsto \mathbb{S}^n = \Sigma^n \mathbb{1} \tag{3}$$

into the Picard group of $\mathscr{C}$ (cf. [MS16, Example 2.1.5]).

Example 3.11 In the symmetric monoidal ∞-category of spectra, the invertible objects are precisely the shifts $\mathbb{S}^d$ of the sphere spectrum, and the map (3) is an isomorphism. The spectrum $\mathbb{S}^1$ is thus characterised by the fact that it generates the Picard group of Sp. Proofs of this (elementary) fact can be found in [HMS94, § 1], [Str92, Theorem 2.2], and [MS16, Example 2.1.6].

Example 3.12 Let G be a finite group and consider the ∞-category $\mathrm{Fun}(BG, \mathrm{Sp})$ of spectra with a G-action, equipped with the pointwise symmetric monoidal structure. The underlying object of an invertible object in this category should be an invertible spectrum, and is thus of the form $\mathbb{S}^d$ for some $d \in \mathbb{Z}$. Moreover, any group must act by equivalences; by the degree isomorphism $\pi_d(\mathbb{S}^d) \simeq \mathbb{Z}$, self-maps of $\mathbb{S}^d$ are determined up to homotopy by their degree, and only the maps of degree 1 or -1 are equivalences. Any group homomorphism

$$G \to \{\pm 1\} \hookrightarrow \mathbb{Z} \simeq \pi_d(\mathbb{S}^d)$$

determines a G-action on $\mathbb{S}^d$, so it follows that any G-action on $\mathbb{S}^d$ makes it into an invertible object in $\mathrm{Fun}(BG, \mathrm{Sp})$.

For the symmetric group $G = \Sigma_n$ (for $n \geqslant 2$), there are precisely two maps $\Sigma_n \to \mathbb{Z}/2$, or equivalently, two Σ_n-module structures on $\mathbb{Z}$: the trivial and the sign representation. The invertible objects in $\mathrm{Fun}(B\Sigma_n, \mathrm{Sp})$ are thus the shifts $\mathbb{S}^d$ of the sphere spectrum with either the trivial or the sign representation. It follows that the Picard group is

$$\pi_0(\mathrm{pic}(\mathrm{Fun}(B\Sigma_n, \mathrm{Sp}))) \simeq \mathbb{Z} \oplus \mathbb{Z}/2.$$

Example 3.13 From the previous example, we obtain a description of the invertible objects in the ∞-category $\mathrm{SSeq}(\mathrm{Sp})$ with the levelwise tensor product. Indeed, this category is equivalent to the coproduct of $\mathrm{Fun}(B\Sigma_n, \mathrm{Sp})$ for all $n \geqslant 0$. The symmetric sequences in spectra that are invertible for the levelwise tensor product are then precisely the sequences A of shifts $A(n) \simeq \mathbb{S}^{d_n}$ of the sphere spectrum with trivial or sign representation (except for $n = 0, 1$, in which case there is only the trivial representation).

The Picard group of operads Observe that the suspension $\mathbf{s}\mathbb{E}_\infty$ of the $\mathbb{E}_\infty$-operad in spectra (as defined in Definition 3.2) has invertible terms. In fact, $\mathbf{s}\mathbb{E}_\infty$ is invertible with respect to the levelwise tensor product, with inverse $\mathbf{s}^{-1}\mathbb{E}_\infty$. We expect that the map

$$\mathbb{Z} \mapsto \pi_0(\mathrm{pic}(\mathrm{Opd}^{\mathrm{nu}}(\mathrm{Sp}))), \quad n \mapsto \mathbf{s}^n\mathbb{E}_\infty \tag{4}$$

is an isomorphism, and in particular that $\mathbf{s}\mathbb{E}_\infty$ generates the Picard group of nonunital operads in spectra with the levelwise tensor product.

Conjecture 3.14 *The Picard group of nonunital operads in spectra is isomorphic to $\mathbb{Z}$ and generated by the suspension $\mathbf{s}\mathbb{E}_\infty$ of the $\mathbb{E}_\infty$-operad.*

As in the case of spectra, this would characterise the operad $\mathbf{s}\mathbb{E}_\infty$. The map (4) is clearly injective (consider the terms in degree two and use the result for spectra); therefore, it remains to show that it is surjective, i.e., that every invertible nonunital operad is equivalent to $\mathbf{s}^d\mathbb{E}_\infty$ for some $d \in \mathbb{Z}$.

Note that the map (4) being an isomorphism means that the existence of an operad structure on a symmetric sequence A should greatly restrict the invertible objects, since we saw in Example 3.13 that the shift d_n of $A(n) \simeq \mathbb{S}^{d_n}$ and the Σ_n-action can be freely chosen, whereas the choice is limited for operads. We claim this is rather plausible: admitting the structure of an operad is indeed quite a strong condition on a symmetric sequence.

Quasi-invertibility We first show that we can detect an operad $\mathbf{O}$ having the same underlying symmetric sequence as a suspension $\mathbf{s}^d\mathbb{E}_\infty$ of the $\mathbb{E}_\infty$-operad by a rather simple criterion: that the terms $\mathbf{O}(n)$ be invertible spectra and the partial composition maps be equivalences. We call operads satisfying these hypotheses *quasi-invertible*.

Definition 3.15 Let $\mathbf{O}$ be a nonunital operad in a symmetric monoidal ∞-category $(\mathscr{C}, \otimes)$. We call $\mathbf{O}$ *quasi-invertible* if the terms $\mathbf{O}(n)$ are invertible objects of $\mathscr{C}$ for all $n \geqslant 1$ and all partial composition maps

$$\circ_i : \mathbf{O}(k) \otimes \mathbf{O}(\ell) \to \mathbf{O}(k + \ell - 1)$$

of $\mathbf{O}$ are equivalences.

Remark 3.16 The condition that an operad $\mathbf{O}$ be quasi-invertible is *a priori* weaker than the existence of what we might call a 'quasi-inverse': an operad $\mathbf{P}$ such that $\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P}$ has symmetric sequence equivalent to $\mathbb{E}_\infty$ and with partial composition maps being equivalences. We claim, however, that being quasi-invertible is in fact equivalent to having a quasi-inverse, and even to being invertible; we outline how one might prove this at the end of the section.

Lemma 3.17 *Let $\mathbf{O}$ be a nonunital operad in spectra. If $\mathbf{O}$ is quasi-invertible, then the underlying symmetric sequence of $\mathbf{O}$ is equivalent to that of $\mathbf{s}^d\mathbb{E}_\infty$ for some $d \in \mathbb{Z}$.*

Proof In low levels, we have $\mathbf{O}(0) = 0$ and $\mathbf{O}(1) = \mathbb{S}$. In the next level, we have a suspended sphere spectrum $\mathbf{O}(2) \simeq \mathbb{S}^d$ for some $d \in \mathbb{Z}$. The partial composition maps give equivalences

$$\mathbf{O}(n+1) = \mathbf{O}(n+2-1) \simeq \mathbf{O}(n) \otimes \mathbf{O}(2) \simeq \mathbf{O}(n) \otimes \mathbb{S}^d$$

for all $n \geqslant 1$. By induction, the operad $\mathbf{O}$ thus has terms $\mathbf{O}(n) \simeq \mathbb{S}^{(n-1)d}$, as does $\mathbf{s}^d\mathbb{E}_\infty$.

It remains to show that the Σ_n-actions of $\mathbf{O}(n)$ and $\mathbf{s}^d\mathbb{E}_\infty(n)$ agree. Desuspending d times, we may assume $d = 0$. For any permutation $\sigma \in \Sigma_n$, the operad structure provides a commutative diagram

$$\begin{array}{ccc} \mathbf{O}(n) \otimes \mathbf{O}(1)^{\otimes n} & \xrightarrow[\simeq]{\sigma_* \otimes \sigma_*} & \mathbf{O}(n) \otimes \mathbf{O}(1)^{\otimes n} \\ & {\scriptstyle \circ,\,\simeq}\searrow \quad \swarrow{\scriptstyle \circ,\,\simeq} & \\ & \mathbf{O}(n) & \end{array}$$

where the self-equivalence σ_* of $\mathbf{O}(1)^{\otimes n}$, permuting the factors by σ, is equivalent to the identity of $\mathbb{S}$. It follows (on account of the degree of self-maps of the sphere spectrum) that

σ_* on $\mathbf{O}(n)$ is also equivalent to the identity of $\mathbb{S}$, and we conclude that Σ_n acts trivially on $\mathbf{O}(n)$, as it does on $\mathbb{E}_\infty(n)$. □

The criterion of quasi-invertibility is in particular satisfied by operads that are invertible for the levelwise tensor product.

Proposition 3.18 *Let $\mathbf{O}$ be a nonunital operad in spectra. If $\mathbf{O}$ is invertible with respect to the levelwise tensor product, then it is quasi-invertible.*

The proof depends on the following observation.

Lemma 3.19 *The restriction*

$$\otimes : \operatorname{Pic}(\mathrm{Sp}) \times \operatorname{Pic}(\mathrm{Sp}) \to \operatorname{Pic}(\mathrm{Sp})$$

of the smash product of spectra to the full subcategory of invertible spectra is conservative.

Proof Let $f : \mathbb{S}^d \to \mathbb{S}^{d'}$ and $g : \mathbb{S}^e \to \mathbb{S}^{e'}$ be maps between invertible spectra such that their smash product $f \otimes g : \mathbb{S}^{d+e} \to \mathbb{S}^{d'+e'}$ is an equivalence. We have to show that f and g are equivalences. By suspending as we please, we may assume that $d' = e' = 0$. Note that $f \otimes g$ can then only be an equivalence if $d = -e$. If $d < 0$, then f is null, and dually, g is null if $d > 0$; thus, we must have $d = 0$ for $f \otimes g$ to be an equivalence. Now f and g are self-maps of the sphere spectrum, and we can argue using the degree isomorphism $\pi_0(\mathbb{S}) \simeq \mathbb{Z}$. The degrees of f and g satisfy

$$1 = \deg(\mathrm{id}_{\mathbb{S}}) = \deg(f \otimes g) = \deg(f)\deg(g),$$

so both degrees are ± 1. We conclude that f and g are equivalences. □

Proof (of Proposition 3.18) Choose an operad $\mathbf{P}$ with $\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P} \simeq \mathbb{E}_\infty$. Then for all $n \geqslant 1$, we have $\mathbf{O}(n) \otimes \mathbf{P}(n) \simeq \mathbb{S}$, so $\mathbf{O}$ has invertible terms. It follows that there exist $d_n \in \mathbb{Z}$ with $\mathbf{O}(n) \simeq \mathbb{S}^{d_n}$ and $\mathbf{P}(n) \simeq \mathbb{S}^{-d_n}$. The partial composition map

$$\circ_i : (\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P})(k) \otimes (\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P})(\ell) \to (\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P})(k + \ell - 1)$$

is the tensor product of the corresponding partial composition maps of $\mathbf{O}$ and $\mathbf{P}$. Thus, it is a tensor product of maps between invertible spectra which is equivalent to the identity of $\mathbb{S}$ (the corresponding structure map of $\mathbb{E}_\infty$). By Lemma 3.19, we conclude that the partial composition map of $\mathbf{O}$ is an equivalence. □

Corollary 3.20 *Let $\mathbf{O}$ be a nonunital operad in spectra. If $\mathbf{O}$ is invertible with respect to the levelwise tensor product, then the underlying symmetric sequence of $\mathbf{O}$ is equivalent to that of $\mathbf{s}^d\mathbb{E}_\infty$ for some $d \in \mathbb{Z}$.*

Proof Directly from Lemma 3.17 and Proposition 3.18. □

Remark 3.21 To conclude that the Picard group of nonunital operads in spectra is isomorphic to $\mathbb{Z}$ and generated by $\mathbf{s}\mathbb{E}_\infty$, it now remains to show for every $d \in \mathbb{Z}$ that the operad structure of $\mathbf{s}^d\mathbb{E}_\infty$ is (up to equivalence) the unique quasi-invertible operad structure on its underlying symmetric sequence. It suffices to show this for $d = 0$, that is, to show that a quasi-invertible operad $\mathbf{O}$ that looks like $\mathbb{E}_\infty$ as symmetric sequences is in fact equivalent to it as an operad. The difficulty here is in producing a comparison map, of which we can try to prove that is an equivalence: given abstract equivalences $\mathbf{O}(n) \simeq \mathbb{S}$, one should construct a large structure of cohering homotopies and higher homotopies. One might attempt such by some inductive argument, but we do not pursue this matter here.

Remark 3.22 In the linear case, say of nonunital operads in the ∞-category $\mathrm{Ch}_{\geqslant 0}(\mathbb{Z})$ of nonnegatively graded chain complexes of $\mathbb{Z}$-modules, it is more straightforward to produce a comparison map. Indeed, let $\mathbf{O}$ be a nonunital operad in $\mathrm{Ch}_{\geqslant 0}(\mathbb{Z})$ and assume that it is quasi-invertible up to quasi-isomorphism: there are isomorphisms of chain complexes

$$H_*(\mathbf{O}(n)) \simeq \mathbb{Z}[0]$$

for all $n \geqslant 1$ and the partial composition maps of $\mathbf{O}$ induce isomorphisms

$$H_*(\circ_i) : H_*(\mathbf{O}(k) \otimes \mathbf{O}(\ell)) \xrightarrow{\simeq} H_*(\mathbf{O}(k + \ell - 1))$$

on homology. Then the symmetric monoidal endofunctor $H_0 : \mathrm{Ch}_{\geqslant 0}(\mathbb{Z}) \to \mathrm{Ch}_{\geqslant 0}(\mathbb{Z})$, sending a chain complex to its 0th homology seen as a chain complex concentrated in degree zero, induces a functor at the level of operads, and we have an isomorphism $H_0(\mathbf{O}) \simeq \mathbb{E}_\infty$. Moreover, there is a natural transformation $\mathrm{id} \to H_0$ from the identity to this endofunctor, and one verifies that this provides an equivalence

$$\mathbf{O} \xrightarrow{\simeq} H_0(\mathbf{O}) \simeq \mathbb{E}_\infty.$$

It follows that the Picard group of nonunital operads in $\mathrm{Ch}_{\geqslant 0}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ via the map $n \mapsto \mathbf{s}^n\mathbb{E}_\infty$.

An analogous strategy fails for the case of spectra, as the 0-truncation of the sphere spectrum $\mathbb{S}$ is the Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbb{Z}$, not the sphere spectrum itself. One could attempt to inductively construct a map, but we do not investigate this method here.

From quasi-invertibility to invertibility We have seen in Proposition 3.18 that for nonunital operads in spectra, the condition of invertibility implies quasi-invertibility. One might wonder about the inverse implication, whether a quasi-invertible operad of spectra must be invertible (or otherwise, to describe the operads that are quasi-invertible but not invertible). Indeed, we expect invertibility and quasi-invertibility to be equivalent conditions for operads in spectra. We sketch how one might go about proving this; we claim one should be able, given a quasi-invertible operad $\mathbf{O}$ in spectra, to produce an operad $\mathbf{P}$ such that (1) $\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P}$ is quasi-invertible and has symmetric sequence equivalent to that of $\mathbb{E}_\infty$, but moreover that (2) $\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P}$ is actually equivalent to $\mathbb{E}_\infty$. The obstacle of Remark 3.21 to producing an equivalence $\mathbf{O} \otimes_{\mathrm{lev}} \mathbf{P} \simeq \mathbb{E}_\infty$ would remain if only the first condition is satisfied, but we claim one should be able to construct the operad $\mathbf{P}$ in such a way that the equivalences $\mathbf{O}(n) \otimes \mathbf{P}(n) \simeq \mathbb{S}$ are sufficiently natural.

Suppose $\mathbf{O}$ is a quasi-invertible nonunital operad in spectra. Then its terms $\mathbf{O}(n)$ are invertible spectra, so in particular dualisable. Applying the Spanier–Whitehead dual

$$(-)^\vee = \mathrm{map}(-, \mathbb{S}) : \mathrm{Sp}^{\mathrm{op}} \to \mathrm{Sp}$$

levelwise to $\mathbf{O}$, one obtains a *co*operad $\mathbf{O}^\vee$. The terms of $\mathbf{O}$ and $\mathbf{O}^\vee$ are levelwise inverse to each other, and the partial composition maps of $\mathbf{O}^\vee$ are equivalences; in other words, $\mathbf{O}^\vee$ is a quasi-invertible cooperad. One would like to invert the structure maps of $\mathbf{O}^\vee$ in a coherent way to make it into an operad.

Observe that in the 1-categorical case, a cooperad having isomorphisms as partial composition maps means one can choose inverses of the structure maps and show that they assemble into the structure of an operad. For operads in higher categories, this is not as straightforward: if the partial composition maps of an operad $\mathbf{O}$ are equivalences, the multiplication map $\mathbf{O} \circ \mathbf{O} \to \mathbf{O}$ need not be an equivalence itself. Encoding operads as certain presheaves, however, we claim the cooperad $\mathbf{O}^\vee$ can be made into an operad.

Hoffbeck–Moerdijk [HM24] define nonunital operads as presheaves (with some additional data) on a category of rooted trees $\mathbb{A}$. The category $\mathbb{A}$ is a (non-full) subcategory of the category $\mathbf{\Omega}$ defined by Moerdijk–Weiss [MW07] in the context of dendroidal sets. The category $\mathbf{\Omega}$ contains the simplex category $\mathbf{\Delta}$, and the relation between categories and simplicial sets extends to a relation between operads (in a more general sense than we use here, namely coloured symmetric operads) and dendroidal sets. Restricting to the subcategory $\mathbb{A}$ of [HM24] imposes the condition that the operads be nonunital.

Informally, one can associate to a nonunital operad $\mathbf{P}$ in a symmetric monoidal ∞-category a $\mathscr{C}$-valued presheaf $F_{\mathbf{P}}$ on $\mathbb{A}$ whose value on a tree T is the tensor product ranging over all the vertices $v \in T$ of the objects $\mathbf{P}(\text{in } v)$, where in v denotes the number of incoming edges of vertex v. The morphisms in $\mathbb{A}^{\text{op}}$ are 'inner edge contractions', and they are sent by $F_{\mathbf{P}}$ to the corresponding composition and tensor product of partial composition maps. Conversely, to let such presheaves F on $\mathbb{A}$ correspond to operads, one has to equip it with more data: for trees S and T and a leaf $v \in S$, there should be a chosen equivalence $F(S \circ_v T) \simeq F(S) \otimes F(T)$ (natural in a certain way), where $S \circ_v T$ denotes the tree obtained by grafting T on top of S at the place of v. These data should moreover satisfy some associativity axioms. We refer to [HM24] for detailed definitions.

This encoding of operads is well-equipped to deal with (co)operads whose partial composition maps are equivalences: if $\mathbf{P}$ is such an operad, the presheaf $F_{\mathbf{P}} : \mathbb{A}^{\text{op}} \to \text{Sp}$ factors through the core $\text{Sp}^{\simeq}$, and the converse is also true. Now one can exploit the fact that any ∞-groupoid, such as the core, is equivalent to its own opposite ∞-category. Alternatively, instead of applying the core $(-)^{\simeq}$ to the codomain, we can apply its left adjoint K of 'groupoid completion' to $\mathbb{A}$, and look at presheaves on $K(\mathbb{A})$. Precomposing with the equivalence $K(\mathbb{A}) \simeq K(\mathbb{A})^{\text{op}}$ then turns a presheaf on $K(A)$ into a copresheaf, and this procedure turns $F_{\mathbf{P}}$ into a cooperad. An equivalence between the ∞-categories of operads in Sp and the ∞-category of certain presheaves on $\mathbb{A}$, and dually, between cooperads and certain copresheaves, would thus equip a quasi-invertible cooperad with the structure of an operad.

We would then obtain an operad structure on $\mathbf{O}^\vee$ exhibiting it as a quasi-inverse to $\mathbf{O}$ (in the sense of Remark 3.16), showing that being quasi-invertible implies having a quasi-inverse. Showing that $\mathbf{O}^\vee$ is an actual inverse now comes down to constructing a comparison map between $\mathbf{O} \otimes_{\text{lev}} \mathbf{O}^\vee$ and $\mathbb{E}_\infty$ and showing it is an equivalence, as in Remark 3.21.

We should, however, be able to exploit the model of operads as presheaves on $\mathbb{A}$ to obtain a stronger result. For any ∞-groupoid K, equipping the functor category $\text{Fun}(K, \text{Sp})$ with the pointwise symmetric monoidal structure (which is closed) should give an equivalence

$$\text{Pic}(\text{Fun}(K, \text{Sp})) \simeq \text{Fun}(K, \text{Pic}(\text{Sp})).$$

In other words, a functor $F : K \to \text{Sp}$ should be invertible with respect to the pointwise tensor product if and only if the spectra $F(x)$ are invertible for all $x \in K$. Then, the presheaf $F_{\mathbf{O}}$ on $K(\mathbb{A})$ associated to a quasi-invertible operad $\mathbf{O}$ is invertible as an object of $\mathrm{Fun}(K(\mathbb{A})^{\mathrm{op}}, \mathrm{Sp})$, and its inverse G should correspond to an inverse of the operad $\mathbf{O}$. Indeed, G being inverse to F in the presheaf category means there is a natural equivalence $F \otimes_{\mathrm{lev}} G \simeq \mathrm{const}_{\mathbb{S}}$, giving rise to an equivalence of operads.

4 Unstable operadic suspension

In the previous section, we discussed operadic suspension for operads in stable ∞-categories $\mathscr{C}$, and in particular in the ∞-category of spectra. The notion should already exist when $\mathscr{C}$ is only assumed to be pointed, specifically in the symmetric monoidal ∞-category Spc_* of pointed spaces with smash product. Indeed, Arone–Kankaanrinta [AK14] and Ching–Salvatore [CS22] construct 'sphere operads' in the 1-category of pointed topological spaces, providing unstable versions of the suspension $\mathbf{s}\mathbb{E}_\infty = \mathrm{End}(\mathbb{S}^{-1})$ of the $\mathbb{E}_\infty$-operad in spectra (the monoidal unit with respect to the levelwise tensor product). These constructions, although not isomorphic, both become equivalent to the spectral operad $\mathbf{s}\mathbb{E}_\infty$ after applying the suspension spectrum functor $\Sigma^\infty : \mathrm{Spc}_* \to \mathrm{Sp}$ levelwise; in other words, they are stably equivalent.

A 'sphere operad' is an operad $\mathbf{S}$ in pointed spaces with terms $\mathbf{S}(n) \simeq S^{n-1}$ for all $n \geqslant 1$ and with partial composition maps that are equivalences; we describe the construction of the operad structure in detail in § 4.1. Smashing an operad $\mathbf{O}$ levelwise with such a sphere operad $\mathbf{S}$ then gives an operad $\mathbf{O} \wedge \mathbf{S}$ with terms

$$(\mathbf{O} \wedge \mathbf{S})(n) = \mathbf{O}(n) \wedge \mathbf{S}(n) \simeq \mathbf{O}(n) \wedge S^{n-1} \simeq \Sigma^{n-1}\mathbf{O}(n),$$

providing an operadic suspension $\mathbf{sO} := \mathbf{O} \wedge \mathbf{S}$ of $\mathbf{O}$ in the unstable case.

As in the stable case, it is desirable to characterise such sphere operads unstably, providing comparisons between the different constructions in the literature and with the stable operadic suspension. In this section, we first prove that the constructions of the sphere operads of Arone–Kankaanrinta and Ching–Salvatore are indeed already unstably equivalent. The proof we present here is by 'point-set' means, that is, by constructing an explicit zigzag of weak equivalences, and not via some characterising universal property (which would be preferable). We conclude with a discussion of to what extent the methods of § 3 for the stable case should or could extend to the unstable case.

4.1 Sphere operads

The existence of homeomorphisms

$$S^{n-1} \wedge S^{k_1-1} \wedge \ldots \wedge S^{k_n-1} \xrightarrow{\cong} S^{k_1+\cdots+k_n-1}$$

for positive integers $n, k_1, \ldots, k_n$ suggests that these homeomorphisms could be chosen in such a way that they fit together as composition maps in the structure of a nonunital operad $\mathbf{S}$ in the 1-category of pointed spaces with $\mathbf{S}(n) = S^{n-1}$ for all $n \geqslant 1$. Indeed, Arone–Kankaanrinta construct such a sphere operad in [AK14] and show that it satisfies some desirable properties (for instance, the structure maps are homeomorphisms). Similarly, Ching–Salvatore construct in [CS22] an operad whose nth space is homotopy equivalent, but not homeomorphic, to the sphere S^{n-1}. Both Arone–Kankaanrinta and

Ching–Salvatore show that their sphere operads are stably equivalent to the coendomorphism operad of the suspended sphere spectrum $\mathbb{S}^1$; thus, the two different constructions produce equivalent operads when passed to spectra. In this section, we prove by 'point-set' means, that is, using explicit constructions, that the sphere operads of Arone–Kankaanrinta and Ching–Salvatore are already unstably equivalent, as operads in pointed spaces.

The Arone–Kankaanrinta sphere operad Arone–Kankaanrinta define sphere operads $\mathbf{S}_p$ for all integers $1 \leqslant p \leqslant \infty$ as the levelwise one-point compactification of 'simplex operads' $\mathbf{\Delta}_p$ with multiplicative structure maps. Explicitly, the pth simplex operad $\mathbf{\Delta}_p$ consists of the spaces

$$\mathbf{\Delta}_p(n) := \Delta_p^{n-1} = \{\, t \in (0,\infty)^n \mid \|t\|_p = 1 \,\},$$

where $\|-\|_p$ denotes the ℓ_p norm; on this space, Σ_n acts by permutation of the coordinates. For $1 \leqslant i \leqslant n$, the ith partial composition map $\cdot_i : \mathbf{\Delta}_p(n) \times \mathbf{\Delta}_p(k) \to \mathbf{\Delta}_p(n+k-1)$ is given by

$$t \cdot_i u := (t_1, \dots, t_{i-1}, t_i u_1, \dots, t_i u_k, t_{i+1}, \dots, t_n). \tag{5}$$

Arone–Kankaanrinta show that the sphere operads $\mathbf{S}_p$ are pairwise weakly equivalent for different p. Of special interest is the operad $\mathbf{S}_\infty$, satisfying an additional desirable property that is not satisfied by the $\mathbf{S}_p$ for $p < \infty$. To compare Arone–Kankaanrinta's operads to Ching–Salvatore's, it is most convenient to use the operad $\mathbf{K}$ of [AK14, Remark 4.2], whose definition was proposed to Arone–Kankaanrinta by Ching, which has additive structure maps and is isomorphic to the simplex operad $\mathbf{\Delta}_\infty$.

The operad $\mathbf{K}$ is defined to consist of the spaces

$$\mathbf{K}(n) := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; \min_{1 \leqslant i \leqslant n} x_i = 0 \right\},$$

inheriting the Σ_n-action from $\mathbb{R}^n$, which permutes the axes. For $1 \leqslant i \leqslant n$, the ith composition map $+_i : \mathbf{K}(n) \times \mathbf{K}(k) \to \mathbf{K}(n+k-1)$ is given by

$$x +_i y := (x_1, \dots, x_{i-1}, x_i + y_1, \dots, x_i + y_k, x_{i+1}, \dots, x_n).$$

We obtain a sphere operad by applying one-point compactification levelwise, which is functorial with respect to proper maps and thus a symmetric monoidal functor from the 1-category of locally compact Hausdorff spaces with proper maps to the 1-category of pointed spaces, the former equipped with the cartesian product and the latter with the smash product. We denote this sphere operad by $\mathbf{S}_{\mathrm{AK}}$.

Another description of the Arone–Kankaanrinta sphere operad (up to isomorphism) that will be useful is in terms of representation spheres. This description is perhaps conceptually most favourable, and is used by [HL24]. Let $\rho_n := \mathbb{R}^n/\Delta$ denote the reduced standard representation of Σ_n, the quotient of the standard Σ_n-representation $\mathbb{R}^n$ (where Σ_n acts by permutation of the axes) by the diagonal Δ (which is precisely the subspace of fixed points). We write $[-] : \mathbb{R}^n \to \rho_n$ for the (Σ_n-equivariant) quotient map.

Lemma 4.1 *For every n, the map $\mathbf{K}(n) \to \rho_n$, $x \mapsto [x]$ is a Σ_n-equivariant homeomorphism.*

Proof This map is the composite of the inclusion $\mathbf{K}(n) \to \mathbb{R}^n$ and the quotient $\mathbb{R}^n \to \rho_n$, both of which are equivariant. Define an inverse map by

$$[x] \mapsto (x_1 - \min_i x_i, \dots, x_n - \min_i x_i).$$

This is well-defined: if we have $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that $x_i - y_i = \lambda$ for all i, then

$$x_j - \min_i x_i = y_j + \lambda - \min_i (y_i + \lambda) = y_j - \min_i y_i.$$

It is readily verified these maps are inverse to each other. □

It follows from the lemma that we can describe the operad $\mathbf{K}$ up to homeomorphism as the operad $\boldsymbol{\rho}$ consisting of the reduced standard Σ_n-representations. For $1 \leqslant i \leqslant n$, the ith partial composition map $+_i : \rho_n \times \rho_k \to \rho_{n+k-1}$ is then given in terms of the corresponding structure map of $\mathbf{K}$ by

$$[x] +_i [y] := [x +_i y].$$

We can therefore also describe the Arone–Kankaanrinta sphere operad $\mathbf{S}_{\mathrm{AK}}$ as the one-point compactification of $\boldsymbol{\rho}$, and thus to have representation spheres S^{ρ_n} as terms; we write $\mathbf{S}^{\rho}$ for this description.

The Ching–Salvatore sphere operad Ching–Salvatore define a sphere operad $\mathbf{S}_V$ for every finite-dimensional real vector space V. The nth term of $\mathbf{S}_V$ is homotopy equivalent to a sphere of dimension $(n-1)\dim V$. As noted in [CS22, Remark 2.18], one advantage of their construction with respect to Arone–Kankaanrinta's is that $\mathbf{S}_V$ inherits a $\mathrm{GL}(V)$-action from the action on V. For our purposes, we may disregard the $\mathrm{GL}(V)$-action, and we consider the sphere operad $\mathbf{S}_{\mathbb{R}}$ associated to the one-dimensional vector space $\mathbb{R}$, which we denote $\mathbf{S}_{\mathrm{CS}}$.

We use the description provided in the proof of [CS22, Proposition 2.15] to define the operad $\mathbf{S}_{\mathrm{CS}}$. One starts with an operad $\mathbf{R}$ (denoted $\mathbf{R}_{\mathbb{R}}$ in [CS22]) with terms $\mathbf{R}(n) := \rho_n \times \boldsymbol{\Delta}_1(n)$ formed from the terms of the operads $\boldsymbol{\rho}$ and $\boldsymbol{\Delta}_1$ introduced before. For $1 \leqslant i \leqslant n$, the ith partial composition map $+_i : \mathbf{R}(n) \times \mathbf{R}(k) \to \mathbf{R}(n+k-1)$ is given by

$$([x], t) +_i ([y], u) := ([x +_i ty], t \cdot_i u) \tag{6}$$

where $\cdot_i$ denotes the corresponding structure map of the simplex operad $\boldsymbol{\Delta}_1$ and

$$x +_i ty := (x_1, \dots, x_{i-1}, x_i + t_i y_1, \dots, x_i + t_i y_k, x_{i+1}, \dots, x_n).$$

The projection $\mathbf{R} \to \boldsymbol{\Delta}_1$ is levelwise a trivial vector bundle with fibre ρ_n. The sphere operad $\mathbf{S}_{\mathrm{CS}}$ is now defined by taking $\mathbf{S}_{\mathrm{CS}}(n)$ to be the Thom space of the vector bundle $\mathbf{R}(n) \to \boldsymbol{\Delta}_1(n)$, which can be described as the smash product

$$\mathbf{S}_{\mathrm{CS}}(n) \cong S^{\rho_n} \wedge \boldsymbol{\Delta}_1(n)_+.$$

The partial composition maps of $\mathbf{S}_{\mathrm{CS}}$ are homeomorphisms induced by isomorphisms of vector bundles over isomorphic bases, and make it into an operad (see [CS22, Proposition 2.17]).

Equivalence of the sphere operads We now show that the sphere operads $\mathbf{S}_{\mathrm{AK}}$ of Arone–Kankaanrinta and $\mathbf{S}_{\mathrm{CS}}$ of Ching–Salvatore are weakly equivalent by constructing an operad $\mathbf{W}$ and a zigzag of weak equivalences

$$\mathbf{S}_{\mathrm{AK}} \xrightarrow{\simeq} \mathbf{W} \xleftarrow{\simeq} \mathbf{S}_{\mathrm{CS}}.$$

We combine the ideas of [AK14] and [CS22] to define the operad $\mathbf{W}$. Let $\mathbf{T}$ denote the operad of [AK14, Definition 5.2], consisting of the spaces

$$\mathbf{T}(n) := \{\, t \in (0,\infty)^n \mid 1 \leqslant \|t\|_1 \text{ and } \|t\|_\infty \leqslant 1 \,\},$$

which contain the $\mathbf{\Delta}_p(n)$ as a deformation retract for all $1 \leqslant p \leqslant \infty$. The partial composition maps are defined by the formula (5) for those of $\mathbf{\Delta}_p$.

Definition 4.2 Define an operad $\mathbf{V}$ by the spaces

$$\mathbf{V}(n) := \rho_n \times \mathbf{T}(n)$$

for $n \geqslant 1$, with composition defined by formula (6).

Identifying ρ_n with $\mathbf{K}(n)$ via Lemma 4.1, it is straightforward to check that $\mathbf{V}$ forms a suboperad of the 'overlapping discs operad' $\mathbf{P}_{\mathbb{R}}$ of [CS22, Definition 2.5], which has terms $\mathbf{P}_{\mathbb{R}}(n) = \mathbb{R}^n \times (0,\infty)^n$ with partial composition maps given by formula (6). The projection maps $\mathbf{V}(n) = \rho_n \times \mathbf{T}(n) \to \mathbf{T}(n)$ are trivial vector bundles with fibre $\rho_n = \mathbb{R}^n/\Delta$, and assemble into a map of operads $\mathbf{V} \to \mathbf{T}$. We wish to apply the Thom space construction levelwise to this map to obtain a new operad; some care should be taken, as the Thom space is only functorial with respect to fibrewise proper maps.

Lemma 4.3 *The partial composition maps of* $\mathbf{V}$ *are fibrewise proper maps of vector bundles.*

Proof The fibres are finite-dimensional real vector spaces, so fibrewise properness is equivalent to fibrewise injectivity, and thus to fibrewise vanishing of the kernel. Using the isomorphism $\mathbf{V}(n) \cong \mathbf{K}(n) \times \mathbf{T}(n)$ obtained from Lemma 4.1, let $(x,t) \in \mathbf{K}(n) \times \mathbf{T}(n)$ and $(y,u) \in \mathbf{K}(k) \times \mathbf{T}(k)$, and assume $(x,t) +_i (y,u) = (0, t \cdot_i u)$. We need to show that $x = 0$ and $y = 0$. By assumption, we have $x +_i ty = 0$, so $x_j = 0$ for $j \neq i$, and

$$x_i + t_i y_\ell = 0$$

for all $1 \leqslant \ell \leqslant k$. From the definition of $\mathbf{K}$, we have $\min_\ell y_\ell = 0$, so we can pick m with $y_m = 0$. By the equation above, we then find $x_i = x_i + t_i y_m = 0$. For any ℓ, we now see that $t_i y_\ell = x_i + t_i y_\ell = 0$, and since t_i is positive, we have $y_\ell = 0$. We have now shown that $x = 0$ and $y = 0$, finishing the proof. □

To define their sphere operads, Arone–Kankaanrinta depend on the fact that one-point compactification is a symmetric monoidal functor from the category of locally compact Hausdorff spaces and proper maps (equipped with the cartesian product) to the category of pointed topological spaces with the smash product (see [AK14, § 4]). We will use an analogous statement for the Thom space construction. For trivial real vector bundles $V \times X \to X$ and $W \times Y \to Y$ with locally compact Hausdorff bases, there is a canonical homeomorphism

$$\operatorname{Th} V \wedge \operatorname{Th} W \cong \operatorname{Th}(V \oplus W)$$

between the smash product of Thom spaces and the Thom space of the external direct sum of V and W, which is the cartesian product in the category of real vector bundles. Thus, the Thom space is a symmetric monoidal functor from the category of trivial real vector bundles with locally compact Hausdorff base and fibrewise proper maps to the category of pointed topological spaces.

Applying the Thom space construction levelwise to the operad $\mathbf{V}$, we therefore obtain an operad $\mathbf{W}$ of pointed spaces with terms

$$\mathbf{W}(n) \cong S^{\rho_n} \wedge \mathbf{T}(n)_+.$$

The operad $\mathbf{W}$ contains the Ching–Salvatore sphere operad $\mathbf{S}_{\mathrm{CS}}$, which has terms $\mathbf{S}_{\mathrm{CS}}(n) = S^{\rho_n} \wedge \mathbf{\Delta}_1(n)_+$, as a suboperad. Moreover, since the subspaces $\mathbf{\Delta}_1(n) \hookrightarrow \mathbf{T}(n)$ are deformation retracts, so are the subspaces $\mathbf{S}_{\mathrm{CS}}(n) \hookrightarrow \mathbf{W}(n)$. In other words, we can see the operad $\mathbf{W}$ as a thickening of the Ching–Salvatore sphere operad $\mathbf{S}_{\mathrm{CS}}$, via Arone–Kankaanrinta's thickening $\mathbf{T}$ of the simplex operad $\mathbf{\Delta}_1$. In particular, the inclusions assemble into a weak equivalence of operads $\mathbf{S}_{\mathrm{CS}} \xrightarrow{\simeq} \mathbf{W}$.

For the Arone–Kankaanrinta operad, we essentially exploit the fact that picking out the elements $(1, \dots, 1)$ defines a map of operads $\mathrm{Com} \to \mathbf{\Delta}_\infty$ (as observed in [AK14]). We define a map $\boldsymbol{\rho} \to \mathbf{V}$ levelwise as the product of the identity of $\boldsymbol{\rho}$ and the composite $\mathrm{Com} \to \mathbf{\Delta}_\infty \hookrightarrow \mathbf{T}$. (However, this map is *not* the product in the category of operads; although the symmetric sequence of $\mathbf{V}$ is the product of those of $\boldsymbol{\rho}$ and $\mathbf{T}$, it is not the product as operads.)

Lemma 4.4 *The maps*

$$\rho_n \to \rho_n \times \mathbf{T}(n) = \mathbf{V}(n), \quad x \mapsto (x, (1, \dots, 1))$$

assemble into a map of operads $\boldsymbol{\rho} \to \mathbf{V}$ and induce a map of operads $\mathbf{S}_{\mathrm{AK}} \to \mathbf{W}$ that is levelwise an inclusion of deformation retracts. In particular, it is a weak equivalence.

Proof Denote the map $\rho_n \to \rho_n \times \mathbf{T}(n)$ by i_n. It follows from the definitions that the maps i_n assemble into a map of operads $i : \boldsymbol{\rho} \to \mathbf{V}$. To see that this map induces a map $\mathbf{S}_{\mathrm{AK}} \to \mathbf{W}$, observe that we can view $\mathbf{S}_{\mathrm{AK}}(n) = S^{\rho_n}$ as the Thom space of the trivial vector bundle $\rho_n \to \mathrm{pt}$ over the point. The map i_n is a map of vector bundles over the map $\mathrm{pt} \to \mathbf{T}(n)$ picking out the element $(1, \dots, 1)$, and it is fibrewise injective and an inclusion of a deformation retract. Hence, we obtain a map of operads $\mathbf{S}_{\mathrm{AK}} = \mathbf{S}^{\rho} \to \mathbf{W}$ that is levelwise an inclusion of a deformation retract, and thus a weak equivalence. □

We now have a zigzag of weak equivalences

$$\mathbf{S}_{\mathrm{AK}} \xrightarrow{\simeq} \mathbf{W} \xleftarrow{\simeq} \mathbf{S}_{\mathrm{CS}}$$

connecting the Arone–Kankaanrinta and Ching–Salvatore sphere operads, and conclude therefore:

Theorem 4.5 *The Arone–Kankaanrinta sphere operad $\mathbf{S}_{\mathrm{AK}}$ and the Ching–Salvatore sphere operad $\mathbf{S}_{\mathrm{CS}}$ are weakly equivalent.*

4.2 Characterising unstable operadic suspension

In § 3, we discussed two potential approaches to characterising operadic suspension in the stable case: via the suspension morphism, and via invertibility for the levelwise tensor product. We briefly comment here on the feasibility of these approaches for characterising unstable operadic suspension.

For the unstable case, we have to rule out the latter approach via invertibility: in the symmetric monoidal ∞-category of pointed spaces with smash product, the only invertible object is the monoidal unit S^0. The sphere operad $\mathbf{S}$ (Theorem 4.5 allows us to use either model), the unstable analogue of $\mathbf{s}\mathbb{E}_\infty = \mathrm{End}(\mathbb{S}^{-1})$, has terms $\mathbf{S}(n) \simeq S^{n-1}$ and is thus not invertible with respect to the levelwise smash product. The approach suggested in § 3.3, characterising $\mathbf{s}\mathbb{E}_\infty$ as a generator of the Picard group of nonunital operads, therefore is not feasible for characterising the sphere operad in the unstable case.

There is more hope for the approach via property ⊛ of the suspension morphism, as suggested by Heuts–Land [HL24], although the unstable situation is more complicated. As in the stable case, a reduced monad T on a pointed $\mathscr{C}$ induces a suspension morphism $\sigma_T : T \to \Omega T \Sigma$, and by applying this construction to the reduced monad $\mathrm{free}_{\mathbf{O}}$ associated to a nonunital operad $\mathbf{O}$ in $\mathscr{C}$, we obtain a map $\mathrm{free}_{\mathbf{O}} \to \Omega \mathrm{free}_{\mathbf{O}} \Sigma$.

As in the stable case, we would like to say that this map comes from an operadic suspension morphism $\sigma : \mathbf{O} \to \mathbf{sO}$, induced by a map $\mathbb{E}_\infty \to \mathbf{S}$ to the sphere operad given by Euler classes $S^0 \to S^{\rho_n}$. Unstably, however, the monad $\Omega \mathrm{free}_{\mathbf{O}} \Sigma$ need not be equivalent to $\mathrm{free}_{\mathbf{sO}}$. The unstable analogue of property ⊛ that should characterise a suspension morphism $\sigma : \mathbf{O} \to \mathbf{sO}$ now reads (following [HL24, property (F)]):

⊛′ There is a canonical equivalence $\Sigma \mathrm{free}_{\mathbf{sO}} \simeq \mathrm{free}_{\mathbf{O}} \Sigma$ in $\mathrm{End}(\mathscr{C})$. The adjoint map $\mathrm{free}_{\mathbf{sO}} \to \Omega \mathrm{free}_{\mathbf{O}} \Sigma$ refines to a map of monads making the diagram of monads

$$\begin{array}{ccc} & \mathrm{free}_{\mathbf{O}} & \\ {\scriptstyle \mathrm{free}_\sigma} \swarrow & & \searrow {\scriptstyle \sigma_{\mathrm{free}_{\mathbf{O}}}} \\ \mathrm{free}_{\mathbf{sO}} & \longrightarrow & \Omega \mathrm{free}_{\mathbf{O}} \Sigma \end{array}$$

commute.

Note that the horizontal map in the diagram is no longer an equivalence in the unstable case, so commutativity of the diagram does not imply an equivalence between the maps of monads free_σ and $\sigma_{\mathrm{free}_{\mathbf{O}}}$. This means it will be even less straightforward to see that property ⊛′ is characterising: where in the stable case we would obtain an equivalence $\mathrm{free}_\sigma \simeq \mathrm{free}_{\sigma'}$ for two suspension morphisms σ and σ' (see Remark 3.8), which under some assumptions we could lift to an equivalence $\sigma \simeq \sigma'$ (for instance for the $\mathbb{E}_n$-operads if their suspensions have nilpotent Euler classes), such an argument does not obviously work unstably.

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