Radboud University Nijmegen

Faculty of Science

# Equivariant homotopy theory via orbits

Bachelor's Thesis in Mathematics

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July 2023

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CHAPTER 1

# Introduction

In this Bachelor's thesis, we study equivariant homotopy theory, the homotopy theory of spaces equipped with symmetries given by a group action. We approach homotopy theory in an equivariant setting using Quillen's theory of model structures for abstract homotopy theory. We discuss the classical way to do equivariant homotopy theory and a dual approach, first described by Erdal and Güçlükan İlhan in [7].

In topology, two topological spaces are considered 'the same' when they are *homeomorphic*, that is, if there is a continuous map between them that has a continuous inverse. Correspondingly, the homeomorphisms are the isomorphisms in the category **Top** of topological spaces and continuous maps. To study and differentiate spaces, in algebraic topology, one assigns algebraic invariants (such as homotopy groups or homology) to topological spaces whose computation may show that two spaces cannot be homeomorphic. These invariants often cannot detect all differences between spaces, however, and preserve a weaker notion of equivalence than being homeomorphic.

In homotopy theory, one studies topological spaces up to *homotopy equivalence*. Informally, two continuous maps are *homotopic* if there is a continuous deformation (a homotopy) between them; a continuous map is a homotopy equivalence if it has an inverse up to homotopy. In other words, we are interested in the *homotopy category* **Ho Top**, with the same objects as **Top**, but with homotopy classes of continuous maps as maps, so that spaces are now isomorphic if there is a homotopy equivalence instead of a homeomorphism between them. This notion of equivalence is often weakened even further, to *weak homotopy equivalences*, which are continuous maps inducing isomorphisms on homotopy groups.

Studying objects up to a notion of equivalence weaker than isomorphism happens more generally in mathematics, and tackling this problem in an abstract setting leads to *abstract homotopy theory*. Quillen's model structures are one of the modern tools for abstract homotopy theory (among others; see for example the discussion and historical account of approaches to abstract homotopy theory of [22]). A model structure on a category consists of three distinguished classes of maps: *weak equivalences*, resembling isomorphisms (for example, maps that are sent to isomorphisms by a chosen functor) and providing the weaker notion of equivalence, and *fibrations* and *cofibrations*, which must satisfy a number of axioms. By equipping a category with a model structure, we obtain a notion of homotopy between maps of that category, and this allows the construction of the *homotopy category*, in which the weak equivalences become isomorphisms. We use the theory of model structures in this thesis to define an equivariant homotopy theory, of objects with a group action.

In category theory and the theory of model categories, a recurring theme is *duality*. For example, the notion of a *product* (e.g., the Cartesian product in the category of sets) is dual to that of a *coproduct* (e.g., the disjoint union in sets); both are instances of the more general and also dual notions of *limits* and *colimits*, respectively. In an equivariant setting, there is a duality between the *fixed-point* and *orbit* objects associated to an object with a group action.

Traditionally, equivariant homotopy theory is introduced 'via fixed points', with a model structure on the category of $G$-spaces in which the weak equivalences are created by the $H$-fixed-point functors for all subgroups $H$ of a finite group $G$. This construction is an example of a 'right induced' model structure, since the fixed-point functor is a right adjoint.

It is natural to ask whether there is a dual model structure on the category of $G$-spaces, where the weak equivalences are instead generated by the $H$-orbit functors for all subgroups $H$ of $G$. In 2019, Erdal and Güçlükan İlhan showed in [7] that such an approach 'via orbits' to equivariant homotopy theory is possible, using the theory from [11] of left induced model structures, since the orbit functor is a left adjoint. Left-inducing model structures is technically more involved than the classical approach of right-inducing, and to apply this theory, it is necessary to replace the category of topological spaces by a sufficiently nice category of spaces. For this, Erdal and Güçlükan İlhan use the category of simplicial sets, on which there is a model structure with a homotopy category equivalent to the homotopy category of topological spaces.

In this thesis, we make heavy use of the powerful language of category theory. For one, we need to use categories to discuss model structures *on* categories. Another reason for making use of the abstract concepts of category theory, is that it enables the use of a general theory in many concrete situations. For example, the category of $G$-spaces is 'just' a diagram category of functors from a category whose 'shape' is determined by the group $G$ to the category of topological spaces. Recognising fixed-point and orbit objects as limits and colimits of such diagrams, we may appeal to 'abstract nonsense' (as category theory is lovingly known) to understand how these constructions interact with, for instance, adjoint functors and other limits and colimits. Category theory originated from algebraic topology, where in the 1940s Mac Lane and Eilenberg developed the notion of a category to study functoriality and naturality. Since then, category theory has found applications all throughout mathematics. The author mainly used Riehl's excellent *Category Theory in Context* [21] to become acquainted with category theory; we do not give an introduction to category theory here, but we will often refer to this book for categorical concepts and arguments. Other sources on category theory used in this thesis include [14], [16].

This thesis is structured as follows. In Chapter 2, we introduce Quillen's theory of model structures as an approach to abstract homotopy theory. We also introduce cofibrantly generated model structures, and provide two important examples of model structures: on the categories of topological spaces and chain complexes. In Chapter 3, we discuss simplicial sets, a model structure on the category of simplicial sets and its relation to the homotopy theory of topological spaces. Chapter 4 presents group actions and related constructions in a categorical language. Although much of the theory would go through as stated for infinite discrete groups, we restrict attention to *finite* groups in this thesis. The theory developed in these chapters finally allows us to introduce equivariant homotopy theory in Chapter 5. We discuss the classical approach via fixed points, the dual approach via orbits, and compare them.

CHAPTER 2

# Model categories

In this chapter, we give an introduction to the theory of model categories, first presented by Quillen in 1967 [17]. Model categories are a tool to generalise the homotopy theory of topological spaces, where spaces are studied up to (weak) homotopy equivalence, to other categories; as such, they may be regarded as 'models for homotopy theory'. Three examples of model categories that we study in some detail are topological spaces (in § 2.5), chain complexes (in § 2.6) and simplicial sets (in Chapter 3). In the final chapter, we look at model categories of spaces with a finite group action to define an equivariant homotopy theory.

In a model category, there are three distinguished classes of maps, called *weak equivalences*, *fibrations* and *cofibrations*. Associated to a model category is its homotopy category (discussed in § 2.2), in which the weak equivalences are formally or freely inverted. Defining such a localisation with respect to a class of maps of weak equivalences is possible in a more general case, but it becomes more tractable with the full structure of a model category.

Throughout this thesis, we follow the 'modern' conventions of Hovey's monograph [13]; in particular, we just use 'model category' where Quillen used 'closed model category', we require model categories to admit all small, and not only finite, limits and colimits, and we assume factorisations to be functorial. The main sources for the exposition in this chapter are [13], [20]; other sources include [2], [5], [12].

## 2.1 Model structures

Before we state the definition of a model structure, we introduce some auxiliary definitions.

Recall that the *walking arrow* $\mathbb{2}$ is the category that contains two objects and a single non-identity map the objects. If $\mathcal{C}$ is a category, then the objects of the functor category $\mathcal{C}^{\mathbb{2}}$ are the maps of $\mathcal{C}$, and a map from $f : A \to B$ to $g : X \to Y$ in $\mathcal{C}^{\mathbb{2}}$ is a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd} \tag{2.1}$$

An object $A$ in a category $\mathcal{C}$ is a *retract* of $B$ if there are maps $s : A \to B$ and $r : B \to A$ such that $rs = 1_A$; the map $r$ is called a *retraction*.

**Definition 2.1.1** · A map $f : A \to B$ in a category $\mathcal{C}$ is a *retract* of a map $g : X \to Y$ if $f$ is a retract of $g$ in $\mathcal{C}^2$. Explicitly, this means that there is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & X \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "1_B"'] & Y \arrow[r] & B \end{tikzcd}$$ ◊

**Definition 2.1.2** · A *functorial factorisation* in a category $\mathcal{C}$ is a pair of functors $\lambda, \rho : \mathcal{C}^2 \to \mathcal{C}^2$ such that $f = \rho f \circ \lambda f$ for all $f \in \mathcal{C}^2$. In particular, we have $\operatorname{dom} \lambda f = \operatorname{dom} f$, $\operatorname{cod} \lambda f = \operatorname{dom} \rho f$ and $\operatorname{cod} \rho f = \operatorname{cod} f$. Applying the functors $\lambda$ and $\rho$ to the commutative square (2.1) in $\mathcal{C}$ (which is a map in $\mathcal{C}^2$), we get a commutative diagram of the form

$$\begin{tikzcd} & A \arrow[r] \arrow[dl, "\lambda f"'] \arrow[dd, "f"] & X \arrow[dr, "\lambda g"] \arrow[dd, "g"'] & \\ C \arrow[rrr] \arrow[dr, "\rho f"'] & & & Z \arrow[dl, "\rho g"] \\ & B \arrow[r] & Y & \end{tikzcd}$$ ◊

**Definition 2.1.3** · A map $i : A \to B$ has the *left lifting property* with respect to $p : X \to Y$ if in all solid commutative squares of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

there is a *lift* $h : B \to X$ (dashed) making the triangles commute. In this case, we also say that $p$ has the *right lifting property* with respect to $i$. ◊

**Definition 2.1.4** · A *model structure* on a category $\mathcal{C}$ consists of three distinguished classes of maps of $\mathcal{C}$, *weak equivalences* (sometimes denoted $\xrightarrow{\sim}$), *fibrations* and *cofibrations*, and two functorial factorisations $(\tilde{\chi}, \varphi)$ and $(\chi, \tilde{\varphi})$ in $\mathcal{C}$. Each of these classes of maps should be closed under composition and contain all identity maps. A map which is both a fibration and a weak equivalence is called an *acyclic fibration*, and a map which is both a cofibration and a weak equivalence is called an *acyclic cofibration*.

These classes of maps and factorisations should satisfy the following axioms:

(MC1) *Two-out-of-three*: For all maps $f : A \to B$ and $g : B \to C$, if two of the three maps $f$, $g$ and $g \circ f$ are weak equivalences, then so is the third.

(MC2) *Retracts*: If $f$ is a retract of $g$ and $g$ is a weak equivalence, fibration or cofibration, then so is $f$.

(MC3) *Lifting*: Cofibrations have the left lifting property with respect to acyclic fibrations, and acyclic cofibrations have the left lifting property with respect to fibrations.

(MC4) *Factorisation*: If $f$ is a map in $\mathcal{C}$, then $\tilde{\chi} f$ is an acyclic cofibration, $\varphi f$ is a fibration, $\chi f$ is a cofibration, and $\tilde{\varphi} f$ is an acyclic fibration. In other words, the map $f$ can be (functorially) factored as an acyclic cofibration followed by a fibration, and as a cofibration followed by an acyclic fibration. ◊

**Definition 2.1.5** · A *model category* is a complete and cocomplete category $\mathcal{C}$ with a model structure on $\mathcal{C}$. ◊

**Lemma 2.1.6** · *The weak equivalences in a model category are precisely the maps that can be factored as an acyclic cofibration followed by an acyclic fibration.*

*Proof.* Since the class of weak equivalences is closed under composition, the composition of an acyclic cofibration and an acyclic fibration is a weak equivalence. Conversely, use (MC4) to factor a weak equivalence as an acyclic cofibration followed by a fibration. By the two-out-of-three property (MC1), the fibration is an acyclic fibration. □

**Lemma 2.1.7** · *The classes of weak equivalences, fibrations and cofibrations in a model category contain all isomorphisms.*

*Proof.* If $f : X \to Y$ is an isomorphism, then the retract diagram

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[d, "f"'] \arrow[rr, bend left, "1_X"] & Y \arrow[r, "f^{-1}"'] \arrow[d, "1_Y"] & X \arrow[d, "f"] \\
Y \arrow[r, "1_Y"] \arrow[rr, bend right, "1_Y"'] & Y \arrow[r, "1_Y"] & Y
\end{tikzcd}$$

shows using (MC2) that $f$ is a weak equivalence, fibration and cofibration since the identity $1_Y : Y \to Y$ is. □

**Definition 2.1.8** · An object $X$ of a model category $\mathcal{C}$ is *cofibrant* if the unique map $\emptyset \to X$ from the initial object to $X$ of $\mathcal{C}$ is a cofibration, and $X$ is called *fibrant* if the unique map $X \to *$ from $X$ to the terminal object of $\mathcal{C}$ is a fibration. ◊

As we will see, the fibrant and cofibrant objects of a model category are often better behaved than arbitrary objects. In important examples of model categories, we will sometimes see that the objects are all fibrant (topological spaces) or all cofibrant (simplicial sets).

**Definition 2.1.9** · By factoring the unique map $\emptyset \to X$ for any object $X$ of a model category $\mathcal{C}$ as a cofibration $\emptyset \to QX$ followed by an acyclic fibration $q_X : QX \to X$, we obtain an endofunctor $Q$ on $\mathcal{C}$ that sends an object to a *cofibrant replacement* $QX$, together with a natural weak equivalence $q : Q \overset{\sim}{\Rightarrow} 1_{\mathcal{C}}$. Dually, by factoring the unique map $X \to *$ as an acyclic cofibration $r_X : X \to RX$ followed by a fibration $RX \to *$, we find an endofunctor $R$ on $\mathcal{C}$ sending an object to its *fibrant replacement*, together with a natural weak equivalence $r : 1_{\mathcal{C}} \overset{\sim}{\Rightarrow} R$. ◊

In particular, every object in a model category is weakly equivalent to a cofibrant and a fibrant object. In the homotopy category of a model category, which we will discuss in § 2.2, weak equivalences become isomorphisms, so a cofibrant or fibrant replacement of an object becomes isomorphic to that object in the homotopy category.

We will now give some simple examples of model categories.

*Example* 2.1.10 (model structures on **Set**) · There are exactly *nine* model structures on the category of sets (see [1]). Here we discuss one of them. Take the epimorphisms (surjective maps) as cofibrations, the monomorphisms (injective maps) as fibrations, and all maps as weak equivalences. The two-out-of-three property (MC1) is then trivial.

To check (MC2), if $f : A \to B$ is a retract of $g : X \to Y$, then we have a commutative diagram of the form

$$\begin{tikzcd}
A \arrow[r, tail, "i"'] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & X \arrow[r, two heads, "r"'] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r, tail, "i'"] \arrow[rr, bend right, "1_B"'] & Y \arrow[r, two heads, "r'"] & B
\end{tikzcd}$$

where $i$ and $i'$ are injective, and $r$ and $r'$ are surjective. In the case that $g$ is a weak equivalence, there is nothing to check. If $g$ is a cofibration, then $fr = r'g$ is an epimorphism, and hence $f$ is an epimorphism thus and a cofibration. Finally, if $g$ is a fibration, then $i'f = gi$ is a monomorphism, and hence $f$ is a monomorphism, thus a fibration.

For (MC3), given a lifting problem

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, two heads, "i"'] & X \arrow[d, tail, two heads, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

where $i$ is a cofibration and $p$ is a fibration (both are necessarily weak equivalences), we can define a lift $h : B \to X$ either as the section of $i$ composed with $f$, or as $g$ composed with the retraction of $p$. From the commutativity of the square, it follows that these definitions are in fact equal; commutativity of the triangles with side $h$ follows directly from the properties of the section of $i$ and retraction of $p$.

Finally, for (MC4), as a factorisation of a map $f : A \to B$ (in both cases since all maps are weak equivalences), we can take

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "f"'] & & B \\ & f(A) \arrow[ur, hook] & \end{tikzcd}$$

where $f : A \to f(A)$ is cofibration since any map is surjective on its image, and the inclusion $f(A) \hookrightarrow B$ is injective and thus a fibration. Checking that this factorisation is functorial is straightforward.

The homotopy category of this model structure on **Set**, in which the weak equivalences – in this case, all maps – are inverted, is equivalent to the terminal category $\mathbb{1}$. ◊

*Example* 2.1.11 · Let $\mathcal{C}$ be any complete and cocomplete category. There is a model structure on $\mathcal{C}$ where all maps are fibrations and cofibrations, and where the weak equivalences are the isomorphisms. With the goal of formally inverting weak equivalences in mind, this model structure is not very interesting: the isomorphisms are already invertible, so the resulting homotopy category will be isomorphic to $\mathcal{C}$. ◊

*Remark* 2.1.12 (duality) · If $\mathcal{C}$ is a model category, then there is a model structure on $\mathcal{C}^{\mathrm{op}}$ where the cofibrations of $\mathcal{C}^{\mathrm{op}}$ are the fibrations of $\mathcal{C}$, the fibrations of $\mathcal{C}^{\mathrm{op}}$ are the cofibrations of $\mathcal{C}$, and the weak equivalences of $\mathcal{C}^{\mathrm{op}}$ are the weak equivalences of $\mathcal{C}$. As a consequence, claims about model categories have dual versions, where cofibrations become fibrations and *vice versa*. This observation is very often used when proving results about model categories. ◊

*Example* 2.1.13 · Let $\mathcal{C}$ be a category with an object $A$. Then the *slice category* $\mathcal{C}/A$ of $\mathcal{C}$ *over* $A$ has as objects the maps $x : X \to A$ into $A$, and a map in $\mathcal{C}/A$ from $x : X \to A$ to $y : Y \to A$ is a map $f : X \to Y$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x"'] & & Y \arrow[dl, "y"] \\ & A & \end{tikzcd}$$

commutes. If $\mathcal{C}$ is a model category, then there is a model structure on $\mathcal{C}/A$ (which is also complete and cocomplete if $\mathcal{C}$ is) where a map $f$ from $x : X \to A$ to $y : Y \to A$ is a weak equivalence, cofibration or fibration if $f : X \to Y$ is in $\mathcal{C}$. The model category axioms follow directly from those of $\mathcal{C}$. Dually, the slice category $A/\mathcal{C}$ of $\mathcal{C}$ *under* $A$, whose objects are maps $x : A \to X$ out of $A$, admits a model structure in a similar way. ◊

The following proposition is a useful characterisation of the (acyclic) fibrations and (acyclic) cofibrations. It shows that either of the classes of fibrations and cofibrations is determined by the other together with the class of weak equivalences. The proof uses all model structure axioms, except for the two-out-of-three property.

**Proposition 2.1.14** · *Let $\mathcal{C}$ be a model category.*

*(i) The cofibrations in $\mathcal{C}$ are precisely the maps that have the left lifting property with respect to acyclic fibrations.*

*(ii) The acyclic cofibrations in $\mathcal{C}$ are precisely the maps that have the left lifting property with respect to fibrations.*

*(iii) The fibrations in $\mathcal{C}$ are precisely the maps that have the right lifting property with respect to acyclic cofibrations.*

*(iv) The acyclic fibrations in $\mathcal{C}$ are precisely the maps that have the right lifting property with respect to cofibrations.*

*Proof.* We only prove the first statement; the proof of the second is similar, and the third and fourth follow by duality (Remark 2.1.12) from the first two. Axiom (MC3) says that cofibrations have the left lifting property with respect to acyclic fibrations. Conversely, let $f : A \to B$ be a map with the left lifting property with respect to acyclic fibrations. Factor $f$ using (MC4) as a cofibration $i : A \to C$ followed by an acyclic fibration $p : C \to B$. Since $f$ has the left lifting property with respect to $p$, there is a lift $r : B \to C$ in the following diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[d, "p"] \\ B \arrow[r, "1_B"'] \arrow[ur, dashed, "r"] & B \end{tikzcd}$$

Commutativity of the bottom triangle means that $r$ is a retraction of $p$. Recognising $f$ as a retract of $i$ in the diagram

$$\begin{tikzcd} A \arrow[r, "1_A"'] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & A \arrow[r, "1_A"'] \arrow[d, "i"] & A \arrow[d, "f"] \\ B \arrow[r, "r"] \arrow[rr, bend right, "1_B"'] & Y \arrow[r, "p"] & B \end{tikzcd}$$

it follows from (MC2) that $f$ is a cofibration. ◻

Combined with Lemma 2.1.6, this shows that the cofibrations and acyclic cofibrations (or, dually, the fibrations and acyclic fibrations) entirely determine the model structure. This observation lies at the heart of the theory of *cofibrantly generated model categories*, which we will discuss in § 2.4.

An example of a property of the classes of cofibrations and fibrations that is easy to prove using the characterisation of Proposition 2.1.14 is the following lemma.

**Lemma 2.1.15** · *In a model category $\mathcal{C}$, the cofibrations and acyclic cofibrations are stable under pushouts, and dually, the fibrations and acyclic fibrations are stable under pullbacks.*

*Proof.* We have to show that a pushout of a cofibration along any map is again a cofibration. Suppose $f : X \to Y$ is a cofibration in the pushout square on the left in the following diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, very near end, "\ulcorner"] & Z \arrow[r] \arrow[d, "g"] & A \arrow[d, "p"] \\ Y \arrow[r] \arrow[urr, dashed, bend left=15] & Y \amalg_X Z \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

By Proposition 2.1.14, to show that $g$ is a cofibration, it suffices to show that $g$ has the right lifting property with respect to acyclic fibrations. Attaching a lifting problem given by an acyclic fibration $p : A \to B$ on the right, we find a lift $Y \to A$ (dashed) in the composite diagram. Applying the universal property of the pushout $Y \amalg_X Z$ to this lift and the map $Z \to A$, we find the desired lift $Y \amalg_X Z \to A$ (dotted). The proof for acyclic fibrations is analogous, lifting against fibrations instead of acyclic fibrations. The proofs of the dual statements are dual. □

Although the map obtained from the universal property of the pushout $Y \amalg_X Z$ in the above proof is unique, the lift $Y \amalg_X Z \to A$ need not be unique since the original lift $Y \to A$ may not be.

## 2.2 Homotopy category

A model structure on a category $\mathcal{C}$ allows us to construct the *homotopy category* $\mathbf{Ho}\,\mathcal{C}$ of $\mathcal{C}$. The idea behind the homotopy category is that $\mathbf{Ho}\,\mathcal{C}$ is like $\mathcal{C}$, but all weak equivalences in $\mathcal{C}$ are turned into isomorphisms in $\mathbf{Ho}\,\mathcal{C}$. Using the model structure, we can define a notion of *homotopy* between maps in $\mathcal{C}$, and, mimicking the notions for topological spaces, define a map to be a homotopy equivalence if it has an inverse up to homotopy. The homotopy relation is, however, not very well-behaved on *all* maps; for instance, it fails to be an equivalence relation in general. When we restrict to the maps from cofibrant to fibrant objects, the homotopy relation *does* become an equivalence relation, and homotopy classes of maps may be composed. An important property of maps between fibrant–cofibrant objects, is that such a map is a weak equivalence if and only if it is a homotopy equivalence (the Whitehead theorem for model categories) [13, Proposition 1.2.8].

There are multiple equivalent descriptions of the homotopy category. In Quillen's original work on model categories [17, Definition 1.1.6], following [8, 1.1], the homotopy category of a model category $\mathcal{C}$ is defined as the *localisation* or *category of fractions* $\mathcal{C}[\mathcal{W}^{-1}]$ of $\mathcal{C}$ with respect to the class $\mathcal{W}$ of weak equivalences, together with a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ that satisfies the universal property which we will give in Definition 2.2.1. The objects of this category are the objects of $\mathcal{C}$, and maps are finite zigzags of maps in $\mathcal{C}$ with only weak equivalences pointing backward, subject to some relations (see also [20, p. 15]). In general, this construction does not produce a locally small homotopy category, however.

Quillen proves in [17, Theorem 1′] that the homotopy category $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the category $\pi\mathcal{C}_{cf}$, whose objects are all the objects of $\mathcal{C}$ that are both fibrant and cofibrant, and whose maps $X \to Y$ are homotopy classes of maps $X \to Y$ in $\mathcal{C}$. The localisation functor $\gamma : \mathcal{C} \to \pi\mathcal{C}_{cf}$ for this category is defined by sending objects to fibrant–cofibrant replacements and sending a map $X \to Y$ to the homotopy class of the induced map between the fibrant–cofibrant replacements of $X$ and $Y$. This category *is* locally small if $\mathcal{C}$ is.

The third description of the homotopy category of a model category $\mathcal{C}$ is as the category $\mathbf{Ho}\,\mathcal{C}$ whose objects are the objects of $\mathcal{C}$, and whose maps $X \to Y$ are homotopy classes of maps in $\mathcal{C}$ between fibrant–cofibrant replacements of $X$ and $Y$. In this case, the localisation functor $\gamma : \mathcal{C} \to \mathbf{Ho}\,\mathcal{C}$ to this category is defined as the identity on objects, and by sending a map $X \to Y$ to the homotopy class of the induced map between the fibrant–cofibrant replacements of $X$ and $Y$. This category is again equivalent to the category $\pi\mathcal{C}_{cf}$, and is also locally small if $\mathcal{C}$ is. This definition of the homotopy category is used in [5], [13].

**Definition 2.2.1** · Let $\mathcal{C}$ be a category and $W$ be a class of maps in $\mathcal{C}$. A *localisation* of $\mathcal{C}$ with respect to $W$ is a functor $F : \mathcal{C} \to \mathcal{D}$ satisfying the following conditions:

(i) $F$ takes maps in $W$ to isomorphisms in $\mathcal{D}$; and

(ii) for any functor $G : \mathcal{C} \to \mathcal{E}$ that takes maps in $W$ to isomorphisms in $\mathcal{E}$, there exists a unique functor $H : \mathcal{D} \to \mathcal{E}$ making the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "G"'] & & \mathcal{D} \arrow[dl, dashed, "H"] \\ & \mathcal{E} & \end{tikzcd}$$

commute. ◊

*Remark* 2.2.2 · Equivalently, condition (ii) of Definition 2.2.1 says that there is a bijective correspondence, given by precomposition with $F$, between functors $\mathcal{D} \to \mathcal{E}$ and functors $\mathcal{C} \to \mathcal{E}$ that take maps in $W$ to isomorphisms. ◊

**Theorem 2.2.3** ([5, Theorem 6.2]) · *For a model category $\mathcal{C}$, the functor $\gamma : \mathcal{C} \to \mathbf{Ho}\,\mathcal{C}$ is a localisation of $\mathcal{C}$ with respect to the class of weak equivalences.*

**Definition 2.2.4** · Similar to preservation, reflection and creation of limits, we say a functor $F : \mathcal{C} \to \mathcal{D}$:

- *preserves* weak equivalences if $Ff$ is a weak equivalence in $\mathcal{D}$ whenever $f$ is a weak equivalence in $\mathcal{C}$;
- *reflects* weak equivalences if $f$ is a weak equivalence in $\mathcal{C}$ whenever $Ff$ is a weak equivalence in $\mathcal{D}$;
- *creates* weak equivalences if $f$ is a weak equivalence in $\mathcal{C}$ if and only if $Ff$ is a weak equivalence in $\mathcal{D}$.

We also use this terminology for the analogous definitions for the classes of cofibrations, acyclic cofibrations, fibrations and acyclic fibrations instead of weak equivalences. ◊

**Definition 2.2.5** · A functor $F : \mathcal{C} \to \mathcal{D}$ between model categories is a *homotopical functor* if $F$ preserves weak equivalences. When $\mathcal{D}$ is not assumed to be a model category, such a functor $F$ is *homotopical* if $F$ takes weak equivalences in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. ◊

If $\mathcal{D}$ is not a model category but is complete and cocomplete, the functor $F$ being homotopical in the second sense is equivalent to $F$ preserving weak equivalences (and hence being homotopical in the first sense) when we take the model structure of Example 2.1.11 on $\mathcal{D}$, where the weak equivalences are the isomorphisms. (Using the more general notion of *homotopical categories* of [20, Definition 2.1.1], we might equip any $\mathcal{D}$ with a homotopical structure allowing such a definition (see for example [20, Example 2.1.4]); for model categories, the category $\mathcal{D}$ must be complete and cocomplete, however.)

*Example* 2.2.6 · It follows from the two-out-of-three property (MC1) and the naturality of $q : Q \stackrel{\sim}{\Rightarrow} 1$ that the cofibrant replacement endofunctor $Q$ on a model category is homotopical. Dually, also the fibrant replacement endofunctor $R$ is homotopical. ◊

Property (ii) of the localisation $\gamma : \mathcal{C} \to \mathbf{Ho}\,\mathcal{C}$ now becomes the following statement.

**Corollary 2.2.7** · *If $\mathcal{C}$ is a model category and $F : \mathcal{C} \to \mathcal{D}$ is a homotopical functor, then there is a unique functor $\mathbf{Ho}\,F : \mathbf{Ho}\,\mathcal{C} \to \mathcal{D}$ such that $\mathbf{Ho}\,F \circ \gamma = F$.*

In other words, a homotopical functor factors uniquely through the localisation functor of its domain. Using Remark 2.2.2, we might rephrase this statement: precomposition with $\gamma : \mathcal{C} \to \mathbf{Ho}\,\mathcal{C}$ induces a bijective correspondence between functors $\mathbf{Ho}\,\mathcal{C} \to \mathcal{D}$ and homotopical functors $\mathcal{C} \to \mathcal{D}$ [20, p. 15]. Moreover, this correspondence is of a 2-categorical nature.

**Lemma 2.2.8** ([13, Lemma 1.2.2], [20, Remark 2.1.11]) · *Natural transformations between homotopical functors $\mathcal{C} \to \mathcal{D}$ from a model category $\mathcal{C}$ correspond bijectively to natural transformations between the induced functors $\mathbf{Ho}\,\mathcal{C} \to \mathcal{D}$ of Corollary 2.2.7.*

## 2.3 Derived functors

We now study when adjunctions $F \dashv U$ of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between model categories induce adjunctions and equivalences at the homotopy level. An on-the-nose extension of a functor between model categories to a functor between their homotopy categories, that is, a functor commuting with $F$ and the localisations, is in general not possible. Under certain conditions, it is however possible to approximate such an extension.

**Definition 2.3.1** · An adjunction $F \dashv U$ of a pair of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between model categories is a *Quillen adjunction* if $F$ preserves cofibrations and acyclic cofibrations. In this case, $F$ is called a *left Quillen functor* and $U$ a *right Quillen functor*. ◊

Since the definition of a Quillen adjunction talks about 'left objects', the left adjoint and cofibrations, one might think that we could dualise (Remark 2.1.12) this definition by speaking about the right adjoint and fibrations instead. In fact, this transformation results in an equivalent definition, as the next lemma shows.

**Lemma 2.3.2** · *For an adjunction $F \dashv U$ between model categories, the following are equivalent:*

*(i) $F \dashv U$ is a Quillen adjunction.*

*(ii) $F$ preserves cofibrations and acyclic cofibrations.*

*(iii) $U$ preserves fibrations and acyclic fibrations.*

*(iv) $F$ preserves cofibrations and $U$ preserves fibrations.*

*(v) $F$ preserves acyclic cofibrations and $U$ preserves acyclic fibrations.*

The proof makes use of Proposition 2.1.14 and the following observation.

**Lemma 2.3.3** · *Let $F \dashv U$ be any (not necessarily Quillen) adjunction of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between model categories. If $i$ is a map in $\mathcal{C}$ and $p$ is a map in $\mathcal{D}$, then $i$ has the left lifting property with respect to $Up$ if and only if $Fi$ has the left lifting property with respect to $p$.*

*Proof.* We only show one direction; the other follows by duality. Suppose that the map $Fi : FA \to FB$ has the left lifting property with respect to $p : X \to Y$. Given the lifting problem in $\mathcal{C}$ of the outer square of the right-hand diagram below, applying the adjunction $F \dashv U$, we find a lift $h^\sharp : FB \to X$ in the left-hand commutative diagram in $\mathcal{D}$:

$$\begin{tikzcd} FA \arrow[r, "f^\sharp"] \arrow[d, "Fi"'] & X \arrow[d, "p"] \\ FB \arrow[r, "g^\sharp"'] \arrow[ur, dashed, "h^\sharp"] & Y \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, "f^\flat"] \arrow[d, "i"'] & UX \arrow[d, "Up"] \\ B \arrow[r, "g^\flat"'] \arrow[ur, dashed, "h^\flat"] & UY \end{tikzcd}$$

Applying the adjunction $F \dashv U$ again, we obtain the commutative diagram on the right, showing that $h^\flat : B \to UX$ is a lift. Hence, $i$ has the left lifting property with respect to $Up$. □

*Proof (of Lemma 2.3.2).* By Proposition 2.1.14 and Lemma 2.3.3, the functor $F$ preserves cofibrations if and only if $U$ preserves acyclic fibrations, and $F$ preserves acyclic cofibrations precisely when $U$ preserves fibrations. □

*Example* 2.3.4 · Let $\mathcal{C}$ be a model category and $A$ and $B$ objects of $\mathcal{C}$. Recall from Example 2.1.13 that there is a model structure on the slice category $A/\mathcal{C}$ of $\mathcal{C}$ under $A$, whose objects are maps in $\mathcal{C}$ out of $A$, and where the weak equivalences, cofibrations and fibrations are created by the forgetful functor $A/\mathcal{C} \to \mathcal{C}$ (and similarly for $B$). If $f : A \to B$ is a map in $\mathcal{C}$, then precomposition with $f$ defines a functor $f^* : B/\mathcal{C} \to A/\mathcal{C}$ sending an object $x : B \to X$ to the composite $xf : A \to X$, and with

$$
\begin{tikzcd}
& B \arrow[dl, "x"'] \arrow[dr, "y"] & \\
X \arrow[rr, "g"'] & & Y
\end{tikzcd}
\qquad \mapsto \qquad
\begin{tikzcd}
& A \arrow[d, dotted, "f"] \arrow[ddl, bend right, "xf"'] \arrow[ddr, bend left, "yf"] & \\
& B \arrow[dl, dotted, "x"'] \arrow[dr, dotted, "y"] & \\
X \arrow[rr, "g"'] & & Y
\end{tikzcd}
$$

on maps, mapping the commutative triangle on the left-hand side to the solid commutative triangle on the right-hand side. Conversely, there is a functor $f_! : A/\mathcal{C} \to B/\mathcal{C}$, called the *cobase change functor*, which is given by pushout along $f$ on objects. Explicitly, the image of an object $x : A \to X$ is the pushout map $f_!(x)$ in the diagram

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "x"'] & B \arrow[d, "f_!(x)"] \\
X \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & X \amalg_A B
\end{tikzcd}
$$

A map $g$ from $x : A \to X$ to $y : A \to Y$ is sent to the unique map $f_!(g)$ making the following diagram commute (induced by the universal property of the pushout $X \amalg_A B$):

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "x"] \arrow[dd, bend right, "y"'] & B \arrow[d, "f_!(x)"] \arrow[ddr, bend left, "f_!(y)"] & \\
X \arrow[r] \arrow[d, "g"] \arrow[ur, phantom, "\ulcorner" very near end] & X \amalg_A B \arrow[dr, dashed, "f_!(g)", "\exists!"'] & \\
Y \arrow[rr] & & Y \amalg_A B
\end{tikzcd}
$$

where $X \to X \amalg_A B$ and $Y \to Y \amalg_A B$ are the pushout inclusion maps.

The cobase change functor $f_! : A/\mathcal{C} \to B/\mathcal{C}$ is left adjoint to the precomposition functor $f^* : B/\mathcal{C} \to A/\mathcal{C}$. The adjunction's natural isomorphism

$$
\mathrm{Hom}_{A/\mathcal{C}}(A \xrightarrow{x} X, A \xrightarrow{yf} X) \cong \mathrm{Hom}_{B/\mathcal{C}}(B \xrightarrow{f_!(x)} X \amalg_A B, B \xrightarrow{y} Y)
$$

is given by sending a map $g$ from $x : B \to X$ to $yf : B \to X$ to the unique map $X \amalg_A B \to Y$, induced by the universal property of the pushout $X \amalg_A B$, that makes the diagram

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "x"'] & B \arrow[d, "f_!(x)"] \arrow[ddr, bend left, "y"] & \\
X \arrow[r] \arrow[drr, bend right, "g"'] \arrow[ur, phantom, "\ulcorner" very near end] & X \amalg_A B \arrow[dr, dashed, "\exists!"'] & \\
& & Y
\end{tikzcd}
\tag{2.2}
$$

commute. Since $f^*$ by definition preserves fibrations and acyclic fibrations, the adjunction $f_! \dashv f^*$ is a Quillen adjunction by Lemma 2.3.2. Dually, there is a Quillen adjunction $f_* \dashv f^!$ between the slice categories $\mathcal{C}/A$ and $\mathcal{C}/B$ *over* $A$ and $B$, where $f_*$ is given by postcomposition and $f^!$ by pullback along $f$, also called *base change*.

In the special case that $A$ is the initial object $\emptyset$ and $f$ the unique map $\emptyset \to B$, the slice category $\emptyset/\mathcal{C}$ is isomorphic to $\mathcal{C}$, and the adjunction above is a free–forgetful adjunction between $\mathcal{C}$ and $B/\mathcal{C}$, where the free object associated to $X \in \mathcal{C}$ is the coproduct inclusion $\mathrm{inj}_X : X \to X \amalg B$. If we additionally assume that $B$ is the terminal object $*$, the slice category $*/\mathcal{C}$ is known as the *category of pointed objects* of $\mathcal{C}$, denoted $\mathcal{C}_*$, and the left adjoint adds a basepoint to an object. The monad $X \mapsto X \amalg *$ on $\mathcal{C}$ induced by this adjunction is known as the *maybe monad* in computing science. $\lozenge$

**Lemma 2.3.5** (Ken Brown) · *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between model categories. If $F$ takes acyclic cofibrations between cofibrant objects to weak equivalences, then $F$ takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if $F$ takes acyclic fibrations between fibrant objects to weak equivalences, then $F$ takes all weak equivalences between fibrant objects to weak equivalences.*

*Proof.* The proof of the second statement is dual to that of the first. Let $f : X \to Y$ be a weak equivalence between cofibrant objects. Factor the coproduct map $f \amalg 1_Y : X \amalg Y \to Y$ as a cofibration $i : X \amalg Y \to Z$ followed by an acyclic fibration $p : Z \to Y$. Recognising the coproduct $X \amalg Y$ as the pushout of the cofibrations $\emptyset \to X$ and $\emptyset \to Y$,

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y \arrow[d, "\mathrm{inj}_Y"] \\ X \arrow[r, "\mathrm{inj}_X"'] & X \amalg Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

it follows from Lemma 2.1.15 that the injections $\mathrm{inj}_X : X \to X \amalg Y$ and $\mathrm{inj}_Y : Y \to X \amalg Y$, and thus also the composites $i \circ \mathrm{inj}_X : X \to Z$ and $i \circ \mathrm{inj}_Y : Y \to Z$, are cofibrations and that $X \amalg Y$ is cofibrant. Since the map $i : X \amalg Y \to Z$ is a cofibration, the object $Z$ is also cofibrant.

The maps $p : X \to Y$, $pi \circ \mathrm{inj}_X = f : X \to Y$ and $pi \circ \mathrm{inj}_Y = 1_Y : Y \to Y$ are weak equivalences, so it follows from the two-out-of-three property that $i \circ \mathrm{inj}_X : X \to Z$ and $i \circ \mathrm{inj}_Y : Y \to Z$ are too. We thus see that these maps are acyclic cofibrations between cofibrant objects, which $F$ takes to weak equivalences. Applying the two-out-of-three property twice in the commutative diagram

$$\begin{tikzcd} FX \arrow[r, "F(i\circ\mathrm{inj}_X)", "\sim"'] \arrow[dr, "Ff"'] & FZ \arrow[d, "Fp"] & FY \arrow[l, "F(i\circ\mathrm{inj}_Y)"', "\sim"] \arrow[dl, "\sim", "1_{FY}"'] \\ & FY & \end{tikzcd}$$

it follows that $Ff : FX \to FY$ is a weak equivalence. $\square$

Let $\gamma : \mathcal{C} \to \mathbf{Ho}\,\mathcal{C}$ and $\delta : \mathcal{D} \to \mathbf{Ho}\,\mathcal{D}$ be localisations of model categories $\mathcal{C}$ and $\mathcal{D}$, and let $F \dashv U$ be a Quillen adjunction of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$. Then $F$ preserves acyclic cofibrations, and in particular takes acyclic cofibrations between cofibrant objects to weak equivalences. By Ken Brown's lemma 2.3.5, the composite

$$\mathcal{C} \xrightarrow{Q} \mathcal{C}_c \xrightarrow{F} \mathcal{D} \xrightarrow{\delta} \mathbf{Ho}\,\mathcal{D}$$

of the cofibrant replacement functor $Q$ from Definition 2.1.9 (which is homotopical, see Example 2.2.6), the functor $F$ restricted to the full subcategory $\mathcal{C}_c$ of $\mathcal{C}$ on the cofibrant objects and the localisation $\delta : \mathcal{D} \to \mathbf{Ho}\,\mathcal{D}$ is homotopical, that is, sends weak equivalences to isomorphisms. By Corollary 2.2.7 this composite induces a unique functor $\mathbf{Ho}\,\delta FQ : \mathbf{Ho}\,\mathcal{C} \to \mathbf{Ho}\,\mathcal{D}$ such that $\mathbf{Ho}\,\delta FQ \circ \gamma = \delta FQ$. The whiskered composite $\delta Fq : \delta FQ \Rightarrow \delta F$ is then also a natural transformation $\mathbf{Ho}\,\delta FQ \circ \gamma \Rightarrow \delta F$. This almost proves the following proposition.

**Proposition 2.3.6** · *If $F : \mathcal{C} \to \mathcal{D}$ is a left Quillen functor, then it has a total left derived functor $\mathbf{L}F := \mathbf{Ho}\,\delta FQ : \mathbf{Ho}\,\mathcal{C} \to \mathbf{Ho}\,\mathcal{D}$.*

**Definition 2.3.7** · Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between model categories with localisations $\gamma : \mathcal{C} \to \mathbf{Ho}\,\mathcal{C}$ and $\delta : \mathcal{D} \to \mathbf{Ho}\,\mathcal{D}$. When the *right* Kan extension

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{D} \arrow[d, "\delta"] \arrow[dl, phantom, "\Uparrow"] \\ \mathbf{Ho}\,\mathcal{C} \arrow[r, dashed, "\mathbf{L}F"'] & \mathbf{Ho}\,\mathcal{D} \end{tikzcd}$$

of $\delta F$ along $\gamma$ exists, it is called the *total left derived functor* $\mathbf{L}F$ of $F$.

Dually, for a functor $U : \mathcal{D} \to \mathcal{C}$, when the *left* Kan extension

$$\begin{tikzcd} \mathcal{D} \arrow[r, "U"] \arrow[d, "\delta"'] & \mathcal{C} \arrow[d, "\gamma"] \arrow[dl, phantom, "\Downarrow"] \\ \mathbf{Ho}\,\mathcal{D} \arrow[r, dashed, "\mathbf{R}U"'] & \mathbf{Ho}\,\mathcal{C} \end{tikzcd}$$

of $\gamma U$ along $\delta$ exists, it is called the *total right derived functor* $\mathbf{R}U$ of $U$. ◊

Having properly introduced these notions, we finish the proof of above that a left Quillen functor has a total left derived functor.

*Proof (of Proposition 2.3.6 [20, Theorem 2.2.8]).* We show that the functor $\mathbf{Ho}\,\delta FQ : \mathbf{Ho}\,\mathcal{C} \to \mathbf{Ho}\,\mathcal{D}$ and the natural transformation $\delta Fq : \mathbf{Ho}\,\delta FQ \circ \gamma \Rightarrow \delta F$ satisfy the universal property of the right Kan extension in $(\mathbf{Ho}\,\mathcal{D})^{\mathbf{Ho}\,\mathcal{C}}$. By Lemma 2.2.8, we may equivalently check this in the full subcategory of $(\mathbf{Ho}\,\mathcal{D})^{\mathcal{C}}$ on the homotopical functors. Consider a homotopical functor $G : \mathcal{C} \to \mathbf{Ho}\,\mathcal{D}$ and a natural transformation $\alpha : G \Rightarrow \delta F$. Then $Gq : GQ \Rightarrow G$ is a natural isomorphism since $G$ is homotopical and $q : Q \Rightarrow 1_{\mathcal{C}}$ is a natural weak equivalence. The naturality square

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "Gq"'] & \delta F \arrow[d, Rightarrow, "\delta Fq"] \\ GQ \arrow[r, Rightarrow, "\alpha Q"'] & \delta FQ \end{tikzcd}$$

shows that $\alpha$ factors through $\delta FQ$ as

$$G \xRightarrow{(Gq)^{-1}} GQ \xRightarrow{\alpha Q} \delta FQ \xRightarrow{\delta Fq} \delta F.$$

Now suppose $\alpha$ also factors as

$$G \xRightarrow{\beta} \delta FQ \xRightarrow{\delta Fq} \delta F.$$

By Ken Brown's lemma 2.3.5, the functor $F$ is homotopical on the full subcategory $\mathcal{C}_c$ of cofibrant objects. For any object $X$ of $\mathcal{C}$, the map $Fq_{QX} : FQ^2X \to FQX$ is then a weak equivalence and $\delta Fq_{QX}$ an isomorphism. On the cofibrant replacements, $\beta$ must thus agree with $\alpha Q \circ (Gq)^{-1}$. From the naturality square

$$\begin{tikzcd} GQ \arrow[r, Rightarrow, "\beta Q"] \arrow[d, Rightarrow, "Gq"'] & \delta FQ^2 \arrow[d, Rightarrow, "\delta FQq"] \\ G \arrow[r, Rightarrow, "\beta"'] & \delta FQ \end{tikzcd}$$

the uniqueness of $\beta$ follows, since the vertical transformations are natural isomorphisms because $q$ is a natural weak equivalence and the functors $G$ and $\delta FQ$ are homotopical. This finishes the proof that $\mathbf{L}F := \mathbf{Ho}\,\delta FQ : \mathbf{Ho}\,\mathcal{C} \to \mathbf{Ho}\,\mathcal{D}$ is a left derived functor of $F$. □

Of course, the construction above can be dualised to obtain a total right derived functor for a right Quillen functor.

**Proposition 2.3.8** · *If $U : \mathcal{D} \to \mathcal{C}$ is a right Quillen functor, then it has a total right derived functor* $\mathbf{R}U := \mathbf{Ho}\,\gamma UR : \mathbf{Ho}\,\mathcal{D} \to \mathbf{Ho}\,\mathcal{C}$.

Taking total derived functors of a Quillen adjunction induces an adjunction at the level of homotopy categories.

**Proposition 2.3.9** ([13, Lemma 1.3.10]) · *If $F \dashv U$ is a Quillen adjunction of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between model categories, then the total derived functors* $\mathbf{L}F : \mathbf{Ho}\,\mathcal{C} \rightleftarrows \mathbf{Ho}\,\mathcal{D} : \mathbf{R}U$ *form an adjunction* $\mathbf{L}F \dashv \mathbf{R}U$ *at the level of homotopy categories.*

Under stronger assumptions, a Quillen adjunction induces an equivalence of homotopy categories.

**Definition 2.3.10** · A Quillen adjunction $F \dashv U$ of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between model categories is a *Quillen equivalence* if for all cofibrant objects $X$ of $\mathcal{C}$ and all fibrant objects $Y$ of $\mathcal{D}$, a map $f^\sharp : FX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if its adjoint $f^\flat : X \to UY$ is a weak equivalence in $\mathcal{C}$. ◊

**Proposition 2.3.11** ([13, Proposition 1.3.13]) · *Let $F \dashv U$ be a Quillen adjunction of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between model categories. Then the adjunction* $\mathbf{L}F \dashv \mathbf{R}U$ *of Proposition 2.3.9 is an adjoint equivalence of categories, meaning that the adjunction unit and counit are natural isomorphisms, if and only if $F \dashv U$ is a Quillen equivalence.*

In the next chapter, we discuss an important example of a Quillen equivalence between the categories of topological spaces and simplicial sets.

*Example* 2.3.12 ([18, Proposition 2.3]) · Let $f : A \to B$ be a weak equivalence in a model category $\mathcal{C}$. The Quillen adjunction $f_! \dashv f^*$ of Example 2.3.4 between the slice categories $A/\mathcal{C}$ and $B/\mathcal{C}$ is a Quillen equivalence if and only if the pushout of $f$ along any cofibration is a weak equivalence (a model category satisfying this property is called *left proper*).

To see this, first suppose that pushouts of $f$ along cofibrations are weak equivalences, and let $x : A \to X$ be a cofibrant object of $A/\mathcal{C}$ and $y : B \to Y$ be a fibrant object of $B/\mathcal{C}$. Since $1_A : A \to A$ is the initial object of $A/\mathcal{C}$, the map $x : A \to X$ is a cofibration in $\mathcal{C}$, and since the unique map $B \to *$ into the terminal object of $\mathcal{C}$ is terminal in $B/\mathcal{C}$, the object $Y$ is fibrant in $\mathcal{C}$. To show that $f_! \dashv f^*$ is a Quillen equivalence, let $g$ be a map from $x : A \to X$ to $f^*(y) = yf : A \to Y$. Then we need to show that $g : X \to Y$ is a weak equivalence if and only if its adjoint $X \amalg_A B \to Y$ is. In the diagram (2.2), we see that the pushout $X \to X \amalg_A B$ of $f$ along $x$ is a weak equivalence, since $x : A \to X$ is a cofibration. By the two-out-of-three property, then, we see from the diagram that $g : X \to Y$ is a weak equivalence if and only if the dashed map $X \amalg_A B \to Y$ is.

Conversely, suppose that $f_! \dashv f^*$ is a Quillen equivalence and let $i : A \to X$ be a cofibration. We want to show that the pushout of $f$ along $i$ is also a weak equivalence. Since $i$ is a map in $A/\mathcal{C}$ from the initial object $1_A : A \to A$ to $i : A \to X$, the object $i : A \to X$ is cofibrant. The fibrant replacement $r_{X\amalg_A B} : X \amalg_A B \to R(X \amalg_A B)$ of the pushout of $f$ and $i$ is a weak equivalence into a fibrant object. Since $f_! \dashv f^*$ is a Quillen equivalence, its adjoint $X \to R(X \amalg_A B)$ is then also a weak equivalence, and it follows from the two-out-of-three property that the pushout $X \to X \amalg_A B$ of $f$ along $i$ is a weak equivalence.

As an example of a map $f$ which is not a weak equivalence for which the Quillen adjunction $f_! \dashv f^*$ is not a Quillen equivalence, consider the special case of the free–forgetful Quillen adjunction $F \dashv U$ between **Set** with the model structure where the weak equivalences are the isomorphisms, and all maps are fibrations and cofibrations (see Example 2.1.11) and the category $\mathbf{Set}_* = */\mathbf{Set}$ of pointed sets (where $f$ is the unique map $\emptyset \to *$ from the initial (empty) set to the terminal (singleton) set). In both of the model categories **Set** and $\mathbf{Set}_*$, every object is fibrant and cofibrant. Choosing the cofibrant object $*$ in **Set** and the fibrant object $1_* : * \to *$ in $*/\mathbf{Set} = \mathbf{Set}_*$, the adjoint of the isomorphism and hence weak equivalence $1_* : * \longrightarrow U(1_*) = *$ is the coproduct map $1_* \amalg 1_* : * \amalg * \longrightarrow *$, which is not an isomorphism and hence not a weak equivalence. The Quillen adjunction $F = f_! \dashv f^* = U$ is thus not a Quillen equivalence in this case. ◊

## 2.4 Cofibrantly generated model categories

An important class of model categories are the *cofibrantly generated model categories*. In such model categories, the fibrations and cofibrations are defined with respect to (small) sets of maps, rather than (potentially and often proper) classes. The idea is that there are sets of maps (satisfying some conditions) that are called *generating cofibrations* and *generating acyclic cofibrations*, meaning that the fibrations are precisely the maps that have the right lifting property against generating acyclic cofibrations (and similarly for the acyclic fibrations). Another important result that we discuss here is the small object argument, which produces functorial factorisations in categories.

*Notation* 2.4.1 · Let $I$ be a class of maps in a category. We write ${}^{\boxslash}I$ for the class of maps that have the left lifting property against all maps of $I$, and, dually, $I^{\boxslash}$ for the class of maps that have the right lifting property against all maps of $I$. ◊

We first recall some set-theoretical definitions. If $\alpha$ is an ordinal, then an *$\alpha$-sequence* in a cocomplete category $\mathcal{C}$ is a colimit-preserving functor $X : \alpha \longrightarrow \mathcal{C}$ from the ordinal category $\alpha$, which is a transfinite sequence of maps $X_\beta \longrightarrow X_{\beta+1}$ for $\beta + 1 < \alpha$. The induced map $X_0 \longrightarrow \operatorname{colim}_{\beta<\alpha} X_\beta$ is called the *$\alpha$-composite* of the $\alpha$-sequence $X$. If $I$ is a class of maps in $\mathcal{C}$, then an $\alpha$-composite of an $\alpha$-sequence of maps of $I$ for any $\alpha$ is called a *transfinite composition* of maps of $I$.

To introduce small objects, we need two definitions related to ordinals and cardinals. Firstly, the *cofinality* of a cardinal $\kappa$ is the smallest ordinal $\alpha$ such that $\kappa$ can be written as a union of $\alpha$ ordinals $< \kappa$. Secondly, a *regular cardinal* is an infinite cardinal $\kappa$ that equals its own cofinality.

**Definition 2.4.2** · Let $\mathcal{C}$ be a cocomplete category, let $I$ be a class of maps in $\mathcal{C}$ and let $\kappa$ be a regular cardinal. An object $A$ of $\mathcal{C}$ is *$\kappa$-small relative to $I$* if, for all ordinals $\alpha \geqslant \kappa$, the representable functor $\operatorname{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \longrightarrow \mathbf{Set}$ preserves transfinite composition of $\alpha$-sequences $X$ of maps $X_\beta \longrightarrow X_{\beta+1}$ in $I$ for $\beta + 1 < \alpha$. The object $A$ is called *small relative to $I$* when there exists a $\kappa$ such that $A$ is $\kappa$-small relative to $I$. ◊

**Definition 2.4.3** · If $\mathcal{C}$ is a cocomplete category and $I$ is a class of maps of $\mathcal{C}$, then a *relative $I$-cell complex* is a transfinite composition of pushouts of maps of $I$. An object $A$ of $\mathcal{C}$ is an *$I$-cell-complex* if the map $\emptyset \longrightarrow A$ is a relative $I$-cell complex. ◊

The following theorem, known as Quillen's small object argument, produces a functorial factorisation with respect to a class of maps in a category. The argument does not presuppose that the category is equipped with a model structure, making it a useful tool to construct a model structure on a category.

**Theorem 2.4.4** (small object argument [13, Theorem 2.1.14]) · *Let $\mathcal{C}$ be a cocomplete category and let $I$ be a class of maps of $\mathcal{C}$ such that the domains of all maps of $I$ are small relative to the class of relative $I$-cell complexes. Then there is a functorial factorisation $(\lambda, \rho)$ on $\mathcal{C}$ such that for any map $f$ in $\mathcal{C}$, the map $\lambda f$ is a relative $I$-cell complex and the map $\rho f$ has the right lifting property against all maps of $I$.*

**Definition 2.4.5** · A model category $\mathcal{C}$ is *cofibrantly generated* by a set of maps $I$ of *generating cofibrations* and a set of maps $J$ of *generating acyclic cofibrations* if the following conditions are satisfied:

(i) the domains of the generating cofibrations are small relative to the class of relative $I$-cell complexes;

(ii) the domains of the generating acyclic cofibrations are small relative to the class of relative $J$-cell complexes;

(iii) the class of fibrations is $I^{\boxslash}$;

(iv) the class of acyclic fibrations is $J^{\boxslash}$. ◊

By Proposition 2.1.14, the class of cofibrations in a model category cofibrantly generated by such classes $I$ and $J$ is precisely ${}^{\boxslash}(I^{\boxslash})$, and the class of acyclic cofibrations is ${}^{\boxslash}(J^{\boxslash})$. We also have the following result about (acyclic) cofibrations in a cofibrantly generated model category.

**Proposition 2.4.6** ([13, Proposition 2.1.18(b), (e)]) · *Let $\mathcal{C}$ be a model category cofibrantly generated by a set $I$ of generating cofibrations and a set $J$ of generating acyclic cofibrations. Then every cofibration in $\mathcal{C}$ is a retract of a relative $I$-cell complex, and every acyclic cofibration is a retract of a relative $J$-cell complex.*

Spelling out the definitions, every (acyclic) cofibration is thus a retract of a transfinite composition of pushouts of generating (acyclic) cofibrations.

Note that conditions (i) and (ii) entail that the factorisations in a cofibrantly generated model category may be produced the small object argument 2.4.4; the factorisations of the model structure need not coincide with those given by the small object argument, however. Using the small object argument and the theory of cofibrantly generated model categories, it becomes easier to construct model structures on categories of interest (see [13, Theorem 2.1.19]).

*Remark* 2.4.7 · It is possible to dualise the definition above to *fibrantly generated model categories*. In practice, there are few cosmall objects in the categories that are often considered, however. For example, only the empty set and the singleton set are cosmall in **Set** [13, p. 34]. ◊

If a model category $\mathcal{C}$ is cofibrantly generated, then there are induced model structures on categories of diagrams in $\mathcal{C}$, which we will use in Chapter 5 to define model structures for equivariant homotopy theory. More precisely, on the category $\mathcal{C}^{\mathcal{J}}$ of $\mathcal{J}$-shaped diagrams for a small category $\mathcal{J}$ in a cofibrantly generated model category $\mathcal{C}$, there are two dual model structures which are again cofibrantly generated, the *projective* and *injective* model structures.

**Theorem 2.4.8** ([12, Theorem 11.6.1]) · *If $\mathcal{C}$ is a cofibrantly generated model category and $\mathcal{J}$ a small category, then there is a* projective model structure *on the diagram category $\mathcal{C}^{\mathcal{J}}$ where weak equivalences and fibrations are pointwise weak equivalences and fibrations in $\mathcal{C}$. Moreover, the projective model structure is again cofibrantly generated.*

Explicitly, a natural transformation $\alpha : F \Rightarrow G$ from functors $F, G : \mathcal{J} \to \mathcal{C}$ is a weak equivalence or fibration if and only if the component map $\alpha_X : FX \to GX$ is in $\mathcal{C}$ for all objects $X$.

For later purposes, we record here the sets of generating cofibrations and acyclic cofibrations of the projective model structure on the diagram category $\mathcal{C}^{\mathcal{J}}$. For objects $X$ of $\mathcal{C}$ and $j$ of $\mathcal{J}$, there is a functor $\mathrm{Hom}_{\mathcal{J}}(j, -) \otimes X : \mathcal{J} \to \mathcal{C}$, called the *free diagram on $X$ generated at $\alpha$* by [12, Definition 11.5.25], sending an object $k$ of $\mathcal{J}$ to the copower $\coprod_{\mathrm{Hom}_{\mathcal{J}}(j,k)} X$. This construction is also functorial in $X$, and defines a functor $\mathcal{C} \to \mathcal{C}^{\mathcal{J}}$. The generating cofibrations of the projective model structure on $\mathcal{C}^{\mathcal{J}}$ are now the maps (natural transformations of functors $\mathcal{J} \to \mathcal{C}$) of the form

$$\mathrm{Hom}_{\mathcal{J}}(j, -) \otimes f : \mathrm{Hom}_{\mathcal{J}}(j, -) \otimes X \to \mathrm{Hom}_{\mathcal{J}}(j, -) \otimes Y,$$

where $f : X \to Y$ is a generating cofibration of $\mathcal{C}$ and $j$ is an object of $\mathcal{J}$. The generating acyclic cofibrations are such maps where $f$ is a generating acyclic cofibration of $\mathcal{C}$.

For the existence of the injective model structure on $\mathcal{C}^{\mathcal{J}}$, it does not suffice for $\mathcal{C}$ to be cofibrantly generated, however, but we also have to assume that $\mathcal{C}$ is *locally presentable*.

**Definition 2.4.9** · For a regular cardinal $\kappa$, a locally small category $\mathcal{C}$ is *locally $\kappa$-presentable* if it is cocomplete and there is a small full subcategory $\mathcal{S}$ of $\mathcal{C}$ such that:

(i) every object of $\mathcal{C}$ can be written as a colimit of a diagram in $\mathcal{S} \hookrightarrow \mathcal{C}$; and

(ii) for every object $X$ of $\mathcal{S}$, the representable functor $\mathrm{Hom}_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathbf{Set}$ preserves $\kappa$-*filtered colimits*, meaning colimits of a shape $\mathcal{J}$ such that every diagram in $\mathcal{J}$ with $< \kappa$ morphisms has a cone.

Finally, a category is *locally presentable* if it is locally $\kappa$-presentable for some regular cardinal $\kappa$. ◊

**Theorem 2.4.10** ([15, Proposition A.2.8.2]) · *If $\mathcal{C}$ is a cofibrantly generated and locally presentable model category and $\mathcal{J}$ a small category, then there is an* injective model structure *on the diagram category $\mathcal{C}^{\mathcal{J}}$ where weak equivalences and cofibrations are pointwise weak equivalences and cofibrations in $\mathcal{C}$. Moreover, the injective model structure is again cofibrantly generated.*

A model category that is both cofibrantly generated and locally presentable is also called *combinatorial.*

The projective and injective model structures provide other examples of Quillen adjunctions.

*Example* 2.4.11 · Let $\mathcal{C}$ be a model category and $\mathcal{J}$ be a small category. If $\mathcal{C}$ has all $\mathcal{J}$-shaped limits and colimits, then the colimit and limit functors are respectively left and right adjoint to the constant diagram functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Delta" description] & \mathcal{C}^{\mathcal{J}} \arrow[l, bend right, "\mathrm{colim}"', "\perp"] \arrow[l, bend left, "\mathrm{lim}", "\perp"'] \end{tikzcd}$$

The constant diagram functor $\Delta$ sends an object $X$ of $\mathcal{C}$ to the constant diagram at $X$ and a map $f : X \to Y$ in $\mathcal{C}$ to the natural transformation $\alpha : \Delta X \Rightarrow \Delta Y$ where the component is $\alpha_j = f$ at every $j \in \mathcal{J}$. With respect to the projective model structure on $\mathcal{C}^{\mathcal{J}}$ (when it exists), the adjunction $\mathrm{colim}_{\mathcal{J}} \dashv \Delta$ is thus a Quillen adjunction since $\Delta$ preserves fibrations and acyclic fibrations. Dually, the adjunction $\Delta \dashv \lim_{\mathcal{J}}$ is a Quillen adjunction with respect to the injective model structure on $\mathcal{C}^{\mathcal{J}}$ (when it exists). ◊

We are now ready to discuss some concrete and interesting examples of model categories.

## 2.5 Topological spaces

In this section, we discuss a model structure on the category of topological spaces. We first fix some definitions and notation.

Throughout this thesis, we follow the algebraic topologist's convention of restricting the category **Top** to the 'convenient' subcategory of compactly generated weak Hausdorff spaces. An important property of this category of spaces, which the category of *all* topological spaces lacks, is that it is Cartesian closed: the function space functor $(-)^X$ is right adjoint to the product functor $- \times X$. We refer to [13, Definition 2.4.21 and Proposition 2.4.22] for a summary of the topological details.

The *n-disk* $D^n$ is the unit disk in $\mathbb{R}^n$, consisting of all points $x$ with $\|x\| \leqslant 1$. The *n-sphere* $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$, which is the boundary of $D^n$, consisting of all points $x$ with $\|x\| = 1$. For every $n$ there is a boundary inclusion $S^{n-1} \hookrightarrow D^n$, where for $n = 0$ we let $D^0 = \{0\}$ and $S^{-1} = \partial D^0 = \emptyset$.

Recall that the *fundamental group* $\pi_1(X, x_0)$ of a based topological space $(X, x_0)$ consists of homotopy classes of loops, that is, based maps $(S^1, s_0) \to (X, x_0)$ where $s_0 \in S^1$ is some fixed basepoint of the circle. The group multiplication of $\pi_1(X, x_0)$ is induced by the composition of loops. Generalising this definition to higher-dimensional spheres $S^n$, we can define the *nth homotopy group* $\pi_n(X, x_0)$ of the based space $(X, x_0)$ to consist of homotopy classes of based maps $(S^n, s_0) \to (X, x_0)$, where $s_0 \in S^n$ is any fixed basepoint of the $n$-sphere. On these sets, a group structure can be defined, justifying the terminology of the higher homotopy *groups*. When $n = 0$, the notation $\pi_0(X, x_0)$ is used for the *set* of homotopy classes of maps $(S^0, s_0) \to (X, x_0)$, which has no preferred group structure; equivalently, $\pi_0(X, x_0)$ is the set of path components of $X$. Just like the fundamental group, the higher homotopy groups define functors $\pi_n : \mathbf{Top}_* \to \mathbf{Grp}$ (and $\pi_0 : \mathbf{Top}_* \to \mathbf{Set}$). In contrast to the fundamental group, the $n$th homotopy groups are always abelian for $n \geqslant 2$. For the topological details and the definition of the group structure of the higher homotopy groups, the reader is referred to [10, § 4.1] or [13, § 2.4].

**Definition 2.5.1** · A continuous map $f : X \to Y$ is a *weak homotopy equivalence* if the induced map

$$f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for all $n \geqslant 0$ and all $x \in X$. ◊

*Example* 2.5.2 · Every homotopy equivalence, and in particular every homeomorphism, is a weak homotopy equivalence. ◊

**Theorem 2.5.3** ([13, Theorem 2.4.19]) · *There is a cofibrantly generated model structure on the category* **Top** *of topological spaces with the weak homotopy equivalences as the weak equivalences, the boundary inclusions $S^{n-1} \hookrightarrow D^n$ for $n \geqslant 0$ as the generating cofibrations, and the inclusions $1_{D^n} \times 0 : D^n \hookrightarrow D^n \times [0, 1]$, $x \mapsto (x, 0)$ for $n \geqslant 0$ as the generating acyclic cofibrations. With respect to this model structure, every space is fibrant.*

Hovey proves this result for the category of all topological spaces, but it also works for our more convenient category of spaces via [13, Theorems 2.4.23 and 2.4.25], and the model structures on **Top** and the category of all topological spaces are Quillen equivalent.

The fibrations of this model structure are called *Serre fibrations*, which are thus the maps that have the right lifting property with respect to all inclusions $1_{D^n} \times 0 : D^n \hookrightarrow D^n \times [0, 1]$. The homotopy category of this model category is equivalent to the category of *CW-complexes* (certain 'nice' topological spaces, which are cofibrant in this model structure) and homotopy classes of maps.

Recognising the category $\mathbf{Top}_*$ of based topological spaces and basepoint-preserving maps as the slice category $*/\mathbf{Top}$ of **Top** under the (terminal) one-point space $*$, we obtain from Example 2.1.13 also a model structure on $\mathbf{Top}_*$, in which a basepoint-preserving map is a weak equivalences, fibrations or cofibrations if the underlying continuous map is. By Example 2.3.4, there is a Quillen adjunction between these model categories.

*Remark* 2.5.4 · There is another model structure on the category of all topological spaces, where the homotopy equivalences (continuous maps with inverses up to homotopy) are taken as weak equivalences; the fibrations in this model structure are called *Hurewicz fibrations*. The homotopy category of this model category is the category with topological spaces as objects and homotopy classes of maps as maps. This was first shown by Strøm in an article appropriately titled 'The Homotopy Category Is a Homotopy Category' [24]. This model structure is not cofibrantly generated, however [2, Proposition 7.2.5]. ◊

*Example* 2.5.5 · With respect to the model structure on topological spaces where the weak equivalences are homotopy equivalences, examples of homotopical functors are 'homotopy invariants' such as the homotopy groups $\pi_n$ of above or the homology groups $\mathrm{H}_n$ (see Example 2.6.9). ◊

## 2.6 Chain complexes

In this section, we give another example of a model structure, on the category of chain complexes of modules over a ring. Throughout the section, we assume that all rings are associative rings with unit. Furthermore, we assume all modules to be left modules, and all chain complexes to be non-negatively graded.

**Definition 2.6.1** · Let $R$ be a ring. A *left $R$-module* or simply *$R$-module* is an abelian group $A$ with a scalar multiplication $R \times A \to A$ that is compatible with the additions and multiplications on $R$

and $A$, in the sense that we have $(r+s)x = rx + sx$, $r(x+y) = rx + ry$, $1x = x$ and $(rs)x = r(sx)$ for all $r, s \in R$ and $x, y \in A$. A *map of $R$-modules $A \to B$*, also called an *$R$-linear map*, is a group homomorphism $f : A \to B$ which preserves scalar multiplication, meaning that $rf(x) = f(rx)$ for all $r \in R$ and $x \in A$. The category of $R$-modules with these maps is denoted $\mathbf{Mod}_R$. ◊

*Example* 2.6.2 · (i) Every ring $R$ is a module over itself.

(ii) A module over the ring $\mathbb{Z}$ of integers is just an abelian group.

(iii) A left ideal of a ring $R$ is a module over $R$.

(iv) If $k$ is a field, then a $k$-module is the same as a $k$-vector space. ◊

**Definition 2.6.3** · Let $R$ be a ring. A (non-negatively graded) *chain complex* $A_\bullet$ of $R$-modules is a sequence $(A_n)_{n\geqslant 0}$ of $R$-modules and a sequence $(\partial_n : A_n \to A_{n-1})_{n\geqslant 0}$ of *boundary maps* (here $A_{-1} := 0$) such that $\partial_n \circ \partial_{n+1} = 0 : A_{n+1} \to A_{n-1}$ for all $n$:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_4"] \arrow[rr, bend left, "0"] & A_3 \arrow[r, "\partial_3"] \arrow[rr, bend right, "0"'] & A_2 \arrow[r, "\partial_2"] \arrow[rr, bend left, "0"] & A_1 \arrow[r, "\partial_1"] \arrow[rr, bend right, "0"'] & A_0 \arrow[r, "\partial_0"] & 0
\end{tikzcd}$$

A map of chain complexes $f : A_\bullet \to B_\bullet$, called a *chain map*, is a sequence $(f_n : A_n \to B_n)_{n\geqslant 0}$ of maps of $R$-modules that commute with the boundary maps of the two chain complexes:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_4"] & A_3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & A_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & A_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & A_0 \arrow[r, "\partial_0"] \arrow[d, "f_0"] & 0 \arrow[d, "0"] \\
\cdots \arrow[r, "\partial_4"'] & B_3 \arrow[r, "\partial_3"'] & B_2 \arrow[r, "\partial_2"'] & B_1 \arrow[r, "\partial_1"'] & B_0 \arrow[r, "\partial_0"'] & 0
\end{tikzcd}$$

The category with chain complexes of $R$-modules as objects and chain maps as morphisms is denoted $\mathbf{Ch}_R$. ◊

*Example* 2.6.4 · If $A$ is any $R$-module, then we can define a chain complex by putting $A$ in degree $n$, and the zero module everywhere else. The boundary maps are necessarily all zero maps. This chain complex is sometimes denoted $S^n(A)$ and $S^n$ when $A = R$.

We can also define a chain complex by putting $A$ in degrees $n$ and $n-1$ with the boundary map $\partial_n := 1_A$, and zero everywhere else. This chain complex is denoted $D^n(A)$ and $D^n$ if $A = R$. ◊

*Example* 2.6.5 (chain complex from short exact sequence) · A *short exact sequence* of $R$-modules is a diagram of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

such that the image of a map is equal to the kernel of the next. Note that the map $A \to B$ must be injective and the map $B \to C$ surjective. We can turn this short exact sequence into a chain complex by taking $C$ in degree $n$, $B$ in degree $n+1$ and $A$ in degree $n+2$, with the maps of the diagram as the boundary maps, and zero everywhere else. ◊

**Lemma 2.6.6** · *For any chain complex,* $\operatorname{im} \partial_{n+1}$ *is a submodule of* $\ker \partial_n$.

*Proof.* If $x \in A_{n+1}$ in a chain complex $A_\bullet$, then $\partial_n(\partial_{n+1}(x)) = 0$, whence $\partial_{n+1}(x) \in \ker \partial_n$. □

It follows that we can form the quotient module $\ker \partial_n / \operatorname{im} \partial_{n+1}$. This module is of special significance, and has received a name:

**Definition 2.6.7** · The *nth homology group* $\mathrm{H}_n A_\bullet$ of a chain complex $A_\bullet$ is defined as the quotient module

$$\mathrm{H}_n A_\bullet := \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

A chain complex $A_\bullet$ is *acyclic* if $\mathrm{H}_n A_\bullet = 0$, or equivalently if $\ker \partial_n = \operatorname{im} \partial_{n+1}$, for all $n$.

With the mapping $A_\bullet \mapsto \mathrm{H}_n A_\bullet$ on objects of $\mathbf{Ch}_R$, the homology groups define *homology functors* $\mathrm{H}_n : \mathbf{Ch}_R \to \mathbf{Mod}_R$. The $n$th homology functor sends a chain map $f : A_\bullet \to B_\bullet$ to a map $\mathrm{H}_n f : \mathrm{H}_n A_\bullet \to \mathrm{H}_n B_\bullet$ of $R$-modules such that the diagram

$$\begin{tikzcd} \ker \partial_n^A \arrow[r, "\tilde{f}_n"] \arrow[d, two heads] & \ker \partial_n^B \arrow[d, two heads] \\ \mathrm{H}_n A_\bullet \arrow[r, "\mathrm{H}_n f"'] & \mathrm{H}_n B_\bullet \end{tikzcd}$$

commutes, where $\tilde{f}_n : \ker \partial_n^A \to \ker \partial_n^B$ is the restriction of $f_n : A_n \to B_n$ to the kernels of the boundary maps $\partial_n^A : A_n \to A_{n-1}$ and $\partial_n^B : B_n \to B_{n-1}$ (which is well-defined by naturality of $f$), and the vertical maps are the quotient maps $x \mapsto x \bmod \operatorname{im} \partial_{n+1}$. ◊

*Example* 2.6.8 · A chain complex obtained from a short exact sequence by the procedure of Example 2.6.5 is acyclic. ◊

*Example* 2.6.9 (singular homology of topological spaces) · Given a simplicial set $X$, we can construct a simplicial abelian group $\mathbb{Z}[X]$ (a contravariant functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Ab}$; see Remark 3.2.2) where $\mathbb{Z}[X]_n := \mathbb{Z}[X_n] = \bigoplus_{X_n} \mathbb{Z}$ is the free abelian group generated by $X_n$. The face map $\mathbb{Z}[X]_n \to \mathbb{Z}_{n-1}$ is defined by applying the face map $d_i : X_n \to X_{n-1}$ on the generating elements; similarly for the degeneracy maps. Associated to this simplicial abelian group is a chain complex, the *Moore chain complex*, also denoted $\mathbb{Z}[X]$, where the boundary maps $\partial_n : \mathbb{Z}[X_n] \to \mathbb{Z}[X_{n-1}]$ are defined as alternating sums of face maps:

$$\partial_n := \sum_{i=0}^{n} (-1)^i d_i = d_0 - d_1 + d_2 - d_3 + \cdots + (-1)^n d_n.$$

The composite functor

$$\mathbf{Top} \xrightarrow{\mathrm{Sing}} \mathbf{sSet} \xrightarrow{\mathbb{Z}[-]} \mathbf{Ch}_{\mathbb{Z}} \xrightarrow{\mathrm{H}_n} \mathbf{Ab}$$

assigns to a topological space its *nth integral singular homology group*. Simplicial sets and the functor $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ will be discussed in Chapter 3. ◊

**Definition 2.6.10** · An $R$-module $P$ is *projective* if in all solid diagrams of the form

$$\begin{tikzcd} & A \arrow[d, two heads, "f"] \\ P \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

where $f$ is an epimorphism, there is a lift $P \to A$ making the diagram commute. ◊

*Example* 2.6.11 · Every free $R$-module, that is, a module with a basis, is projective. In particular, every vector space over a field $k$ is a projective $k$-module. ◊

**Definition 2.6.12** · The *cokernel* of an $R$-linear map $f : A \to B$ is the quotient module

$$\operatorname{coker} f := B / \operatorname{im} f.$$

In categorical language, the cokernel of $f$ is the coequaliser of $f$ and the zero map $0 : A \to B$. ◊

**Theorem 2.6.13** · *There is a model structure on the category* $\mathbf{Ch}_R$ *of chain complexes where a map* $f : A_\bullet \to B_\bullet$ *is*

- *a weak equivalence if the induced map* $\mathrm{H}_n f : \mathrm{H}_n A_\bullet \to \mathrm{H}_n B_\bullet$ *is an isomorphism for all* $n \geqslant 0$*;*
- *a cofibration if* $f_n : A_n \to B_n$ *is a monomorphism with projective cokernel for all* $n \geqslant 0$*;*
- *a fibration if* $f_n : A_n \to B_n$ *is an epimorphism for all* $n \geqslant 1$.

This model structure is called the *projective model structure* on $\mathbf{Ch}_R$. The theorem is proven in [5, § 7] by directly verifying that the model category axioms hold. This is possible since the chain complexes are assumed to be non-negatively graded, permitting an argument by induction.

CHAPTER 3

# Simplicial sets

This chapter introduces simplicial sets. In § 3.1 and § 3.2, we give the definition of simplicial sets and some examples. In § 3.3 and § 3.4, we define an adjunction of functors between the category of simplicial sets and the category of topological spaces, consisting of the *geometric realisation* of a simplicial set and the *singular set* of a topological space. Our primary motivation for simplicial sets here is the model structure on the category of simplicial sets that we will define in § 3.5. We will see that the adjunction of the geometric realisation and the singular functor is a Quillen equivalence with respect to the model structure on topological spaces of § 2.5, whence it follows that the respective homotopy categories are equivalent. This equivalence makes it possible to say things about the homotopy category of topological spaces by studying simplicial sets, which are in general easier to study because of their combinatorial nature.

In this chapter, we largely follow the presentation of [9] for the definitions and examples related to simplicial sets, and [13] for the model structure on the category of simplicial sets. Another source we used is [19].

## 3.1 The simplex category

**Definition 3.1.1** · The *simplex category* $\mathbf{\Delta}$ has as objects the finite non-empty ordinal numbers $\{0, \dots, n\}$ and as maps the order-preserving maps, that is, maps $f : \{0, \dots, m\} \to \{0, \dots, n\}$ such that $f(i) \leqslant f(j)$ whenever $i \leqslant j$. Equivalently, in the sense of equivalences of categories, since every totally ordered finite set is isomorphic to a finite ordinal, the simplex category might be described as having all totally ordered finite sets as objects, together again with order-preserving maps. Yet another equivalent description of $\mathbf{\Delta}$, which is even isomorphic to the first, is as the category of finite non-empty ordinal categories freely generated by the graphs of the form

$$0 \longrightarrow 1 \longrightarrow \dots \longrightarrow n-1 \longrightarrow n,$$

and functors as maps. We write $\mathbf{n}$ for the object $\{0, \dots, n\}$ of $\mathbf{\Delta}$. ◊

*Remark* 3.1.2 · We use the bold symbol $\mathbf{n}$ for the ordinal category corresponding to the ordinal number $n + 1 = \{0, \dots, n\}$, which will be useful when we relate simplicial sets to topological spaces. ◊

*Remark* 3.1.3 · The specific presentation of the simplex category we use depends on the application. Note that the first and third descriptions, using ordinal numbers and ordinal categories, have the advantage of being small, whereas the second is not. ◊

There are two distinguished classes of maps in the simplex category $\mathbf{\Delta}$: the *coface* maps $d^i : \mathbf{n-1} \to \mathbf{n}$ for $0 \leqslant i \leqslant n$ with the defining property that $d^i$ is injective and $i \in \mathbf{n}$ is not in the image of $d^i$; and the *codegeneracy* maps $s^j : \mathbf{n+1} \to \mathbf{n}$ for $0 \leqslant j \leqslant n$ with the defining property that $s^j$ is surjective and $s^j(j) = s^j(j+1) = j \in \mathbf{n}$. The image of the coface map $d^i$ can be seen as the sequence of maps

$$0 \longrightarrow 1 \longrightarrow \dots \longrightarrow i-1 \longrightarrow i+1 \longrightarrow \dots \longrightarrow n-1 \longrightarrow n.$$

The map $i-1 \to i+1$ here is the composition of the maps $i-1 \to i$ and $i \to i+1$. Similarly, the image of the codegeneracy map $s^j$ can be seen as the sequence of maps

$$0 \longrightarrow 1 \longrightarrow \dots \longrightarrow j \xrightarrow{1_j} j \longrightarrow \dots \longrightarrow n-1 \longrightarrow n.$$

The following lemma shows the importance of these classes of maps.

**Lemma 3.1.4** · *Every map in $\Delta$ can be written as a composition of coface and codegeneracy maps.*

*Proof.* We prove by induction on $m$ and $n$ that every map $f : \mathbf{m} \to \mathbf{n}$ can be written as a composition of coface and codegeneracy maps. The base case $f = 1_{\mathbf{0}} : \mathbf{0} \to \mathbf{0}$ is clear, as the empty composition. In the induction step, let $f : \mathbf{m} \to \mathbf{n}$ be a map in $\Delta$. If $f$ is bijective, then $f$ is the identity since $f$ is a monotone map between finite posets. If $f$ is not injective, then we have $f(j) = f(j')$ for some distinct $j, j' \in \mathbf{m}$, and without loss of generality, we can assume that $j' = j+1$, since the order-preserving map $f$ must be constant on the interval between $j$ and $j'$. Then $f$ can be factored as $f = gs^j$ for some $g : \mathbf{m}-1 \to \mathbf{n}$. Using the induction hypothesis, we get the desired factorisation of $f$. If $f$ is not surjective, then there exists an $i \in \mathbf{n}$ with $i \notin f(\mathbf{m})$. In this case, we can factor $f$ as $f = d^i h$ for some $h : \mathbf{m} \to \mathbf{n-1}$. From the induction hypothesis, we again get the desired factorisation of $f$. □

A useful corollary of this lemma is that to define a functor out of $\Delta$ (or $\Delta^{\mathrm{op}}$), it suffices to define the functor on the coface and codegeneracy maps. It turns out that to check whether such a definition is in fact functorial, it is not necessary to check all possible compositions of coface and codegeneracy maps; the coface and codegeneracy maps generate the simplex category together with the following relations, the *cosimplicial identities*:

$$\begin{cases} d^j d^i = d^i d^{j-1} & \text{if } i < j, \\ s^j d^i = d^i s^{j-1} & \text{if } i < j, \\ s^j d^j = 1 = s^j d^{j+1}, & \\ s^j d^i = d^{i-1} s^j & \text{if } i > j+1, \\ s^j s^i = s^i s^{j+1} & \text{if } i \leqslant j. \end{cases}$$

*Proof (cosimplicial identities).* To prove the second identity, let $i < j$. Then we have:

$$\begin{array}{ccccccccc}
0 & \xmapsto{d^i} & 0 & \xmapsto{s^j} & 0 & \xleftarrow{d^i} & 0 & \xleftarrow{s^{j-1}} & 0 \\
1 & \longmapsto & 1 & \longmapsto & 1 & \longleftarrow & 1 & \longleftarrow & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
i-1 & \longmapsto & i-1 & \longmapsto & i-1 & \longleftarrow & i-1 & \longleftarrow & i-1 \\
i & \longmapsto & i+1 & \longmapsto & i+1 & \longleftarrow & i & \longleftarrow & i \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
j-1 & \longmapsto & j & \longmapsto & j & \longleftarrow & j-1 & \longleftarrow & j-1 \\
j & \longmapsto & j+1 & \longmapsto & j & \longleftarrow & j-1 & \longleftarrow & j \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
n-1 & \longmapsto & n & \longmapsto & n-1 & \longleftarrow & n-2 & \longleftarrow & n-1
\end{array}$$

showing that $s^j d^i = d^i s^{j-1}$. The other identities are proven similarly. □

To make the claim above precise:

**Lemma 3.1.5** · *The simplex category* $\mathbf{\Delta}$ *is equivalent to the category generated by the objects* $\mathbf{n}$*, the coface and codegeneracy maps, and the cosimplicial identities.*

The proof can be found in [8, Lemma 2.2] or [16, § VII.5]. The idea behind the proof is to use the cosimplicial identities to rewrite any composition of coface and codegeneracy maps to a certain canonical form. (Or, to put it in computing science terms, the lemma can be proven by turning the cosimplicial identities into a certain term rewriting system and showing that it is strongly normalising and confluent.)

It follows that a (covariant or contravariant) functor out of $\mathbf{\Delta}$ can be defined by defining it on the objects and the coface and codegeneracy maps and verifying that the images of these maps under the functor satisfy relations corresponding to the cosimplicial identities.

## 3.2 Simplicial sets

**Definition 3.2.1** · A *simplicial set* is a contravariant functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ out of the simplex category $\mathbf{\Delta}$. The category of simplicial sets is the functor category $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$, denoted $\mathbf{sSet}$. Correspondingly, a map $X \to Y$ of simplicial sets, called a *simplicial map*, is a natural transformation $X \Rightarrow Y$.

More explicitly, a simplicial set $X$ consists of a sequence $(X_n \coloneqq X(\mathbf{n}))_{n \geqslant 0}$ of sets. The elements of $X_n$ are called the *n-simplices* of $X$. The images under $X$ of the coface and codegeneracy maps in the simplex category are denoted $d_i \coloneqq X(d^i) : X_n \to X_{n-1}$ and $s_j \coloneqq X(s^j) : X_n \to X_{n+1}$, and they are called the *face* and *degeneracy* maps of $X$. A simplex is called *degenerate* if it is in the image of a degeneracy map.

Similarly, directly from the definition of a natural transformation, a simplicial map $f : X \to Y$ consists of component maps $f_n : X_n \to Y_n$ such that the following diagram commutes for all maps $g : \mathbf{m} \to \mathbf{n}$ in $\mathbf{\Delta}$:

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "Xg"'] & Y_n \arrow[d, "Yg"] \\ X_m \arrow[r, "f_m"'] & Y_m \end{tikzcd} \tag{3.1}$$

◊

*Remark* 3.2.2 · More generally, a *simplicial object* in a category $\mathcal{C}$ is a contravariant functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$. The category of simplicial objects of $\mathcal{C}$ is the functor category $\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$, which is denoted $\mathbf{s}\mathcal{C}$. The maps between simplicial objects and the face and degeneracy maps of a simplicial object are introduced similarly to the case of simplicial sets. Although some results in this chapter generalise to the setting of simplicial objects in an arbitrary category, we will focus here on the concrete case when $\mathcal{C}$ is the category of sets. ◊

Since a simplicial set is a contravariant functor, it preserves the cosimplicial identities, but the order of composition is flipped. The images of the cosimplicial identities under a simplicial set are called the *simplicial identities*:

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j, \\ d_i s_j = s_{j-1} d_i & \text{if } i < j, \\ d_j s_j = 1 = d_{j+1} s_j, & \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1, \\ s_i s_j = s_{j+1} s_i & \text{if } i \leqslant j. \end{cases}$$

As we have seen in Lemma 3.1.5, to define a simplicial set $X$, it suffices to give a family of sets $(X_n)_{n \geqslant 0}$, define the face and degeneracy maps and show that they satisfy the simplicial identities. As Riehl points out in [19], however:

> 'Mercifully, the required relations are often obvious, and even if they are not, it is still advisable to assert that they are, after privately verifying that they do in fact hold.'

Again by Lemma 3.1.5, to verify that a family $(f_n : X_n \to Y_n)_{n\geqslant 0}$ defines a simplicial map $X \to Y$, it suffices to consider the naturality squares (3.1) when $g$ induces a face or degeneracy map.

*Example* 3.2.3 · Given any set $A$, we can define a *discrete* simplicial set $X$ by setting $X_n := A$ in all levels $n$, and taking the identity map for all face and degeneracy maps. It is quite clear that this choice of maps satisfies the simplicial identities. This construction defines a functor $\mathbf{Set} \to \mathbf{sSet}$, which is just the constant diagram functor. ◊

*Example* 3.2.4 (nerve of a category) · The *nerve* of a small category $\mathcal{C}$ is the simplicial set

$$N\mathcal{C} := \mathrm{Hom}_{\mathbf{Cat}}(-, \mathcal{C}) : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set},$$

where we regard $\mathbf{\Delta}$ as the full subcategory of $\mathbf{Cat}$ spanned by the finite non-empty ordinal categories. The $n$-simplices of the nerve $N\mathcal{C}$ are thus the functors $\mathbf{n} \to \mathcal{C}$, which correspond to strings of composable maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n.$$

in $\mathcal{C}$. In particular, the 0-simplices are just objects of $\mathcal{C}$, and a 1-simplex is a single map.

Applying the face map $d_i : N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ for $0 < i < n$ to the $n$-simplex above, we get the $(n-1)$-simplex

$$X_0 \xrightarrow{f_0} \dots \xrightarrow{f_{i-2}} X_{i-1} \xrightarrow{f_i f_{i-1}} X_{i+1} \xrightarrow{f_{i+1}} \dots \xrightarrow{f_{n-1}} X_n.$$

The face maps $d_0$ and $d_n$ drop the first and last map, respectively. The degeneracy map $s_j : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ sends the $n$-simplex to the $(n+1)$-simplex

$$X_0 \xrightarrow{f_0} \dots \xrightarrow{f_{j-1}} X_j \xrightarrow{1_{X_j}} X_j \xrightarrow{f_j} \dots \xrightarrow{f_{n-1}} X_n.$$

For example, we recognise the identity maps of $\mathcal{C}$ as the degenerate 1-simplices.

The construction of the nerve defines a functor $N : \mathbf{Cat} \to \mathbf{sSet}$ from the category of small categories. This functor sends a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories to the simplicial map $NF : N\mathcal{C} \to N\mathcal{D}$ that applies $F$ to the objects and maps in the strings. ◊

*Example* 3.2.5 (nerve of a group) · Viewing a discrete group $G$ as a one-object groupoid $\mathbf{B}G$, the *delooping groupoid* of $G$ – that is, the category with a single object $*$ and a map $g : * \to *$ for every group element $g \in G$, with composition defined by the group's multiplication – we can construct the nerve $N\mathbf{B}G$ of $\mathbf{B}G$, which we will call the *nerve* of $G$ and denote by $NG$. An $n$-simplex is simply a list $(g_1, \dots, g_n)$ of elements of $G$. In particular, there is only one 0-simplex, and the 1-simplices are just the group elements. The face map $d_i : NG_n \to NG_{n-1}$ multiplies the $i$th and the $(i+1)$th elements (or, informally, 'removes the $i$th comma') if $0 < i < n$:

$$(g_1, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_n) \xmapsto{d_i} (g_1, \dots, g_{i-1}, g_i \cdot g_{i+1}, g_{i+2}, \dots, g_n).$$

The face maps $d_0$ and $d_n$ drop respectively the first and the last element of the sequence. The degeneracy map $s_j : NG_n \to NG_{n+1}$ adds the identity element $e \in G$ after the $j$th component:

$$(g_1, \dots, g_j, g_{j+1}, \dots, g_n) \xmapsto{s_j} (g_1, \dots, g_j, e, g_{j+1}, \dots, g_n).$$

The degenerate $n$-simplices are exactly those simplices that contain the identity element. In particular, the identity element is the only degenerate 1-simplex. ◊

*Example* 3.2.6 (truncation and skeleton [21, Example 6.2.12]) · Let $\mathbf{\Delta}_{\leqslant n}$ be the full subcategory of the simplex category $\mathbf{\Delta}$ on the objects $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}$. There is an inclusion functor $i_n : \mathbf{\Delta}_{\leqslant n} \hookrightarrow \mathbf{\Delta}$, and precomposition with $i_n$ defines the *n-truncation* $\mathrm{tr}_n \coloneqq i_n^* : \mathbf{sSet} \to \mathbf{Set}^{\mathbf{\Delta}_{\leqslant n}^{\mathrm{op}}}$. By an *n-truncated simplicial set* we mean a functor $\mathbf{\Delta}_{\leqslant n}^{\mathrm{op}} \to \mathbf{Set}$. The $n$-truncation functor has a fully faithful left adjoint $\mathrm{sk}_n : \mathbf{Set}^{\mathbf{\Delta}_{\leqslant n}^{\mathrm{op}}} \to \mathbf{sSet}$, the *n-skeleton*, which is given by left Kan extension along $i_n$. For an $n$-truncated simplicial set $X$, we have $(\mathrm{sk}_n X)_k = X_k$ for $k \leqslant n$, and the simplices above level $n$ of the $n$-skeleton $\mathrm{sk}_n X$ of $X$ are all degenerate. ◊

*Example* 3.2.7 (graphs as simplicial sets) · Let **Graph** denote the category of directed graphs, possibly with loops (edges from a vertex to itself) and multiple edges between a pair of vertices. Alternatively, **Graph** is the category $\mathbf{Set}^{\mathbf{\Gamma}}$ of functors $\mathbf{\Gamma} \to \mathbf{Set}$ from the category $\mathbf{\Gamma}$ with two objects $V$ and $E$ and two parallel maps $s, t : E \to V$; as such, maps of graphs are natural transformations. The images of $V$ and $E$ under a graph (i.e., a functor $\mathbf{\Gamma} \to \mathbf{Set}$) are the sets of vertices and edges, respectively, often also denoted $V$ and $E$. The images of $s$ and $t$ assign to an edge its source and target.

The category $\mathbf{\Gamma}$ embeds into $\mathbf{\Delta}_{\leqslant 1}^{\mathrm{op}}$ by a functor $i : \mathbf{\Gamma} \hookrightarrow \mathbf{\Delta}_{\leqslant 1}^{\mathrm{op}}$, $s \mapsto d_1$, $t \mapsto d_0$. Precomposition with $i$ defines a forgetful functor $U \coloneqq i^* : \mathbf{Set}^{\mathbf{\Delta}_{\leqslant 1}^{\mathrm{op}}} \to \mathbf{Graph}$ which sends a 1-truncated simplicial set $X$ to the graph whose vertices and edges are respectively the 0- and 1-simplices of $X$, with source and target functions given by $d_1$ and $d_0$. The forgetful functor has a faithful left adjoint $F : \mathbf{Graph} \hookrightarrow \mathbf{Set}^{\mathbf{\Delta}_{\leqslant 1}^{\mathrm{op}}}$ which sends a graph $G$ with vertices $V$ and edges $E$ to the 'free' 1-truncated simplicial set $X$ with 0-simplices $X_0 \coloneqq V$ and 1-simplices $X_1 \coloneqq V \amalg E$, the disjoint union of $V$ and $E$. The face maps of $X$ are defined as $d_0 \coloneqq 1_V \amalg t : V \amalg E \to V$ and $d_1 \coloneqq 1_V \amalg s : V \amalg E \to V$, and the degeneracy map is $s_0 \coloneqq \mathrm{inj}_V : V \to V \amalg E$. Composing the free functor $F : \mathbf{Graph} \hookrightarrow \mathbf{Set}^{\mathbf{\Delta}_{\leqslant 1}^{\mathrm{op}}}$ with the skeleton $\mathrm{sk}_1 : \mathbf{Set}^{\mathbf{\Delta}_{\leqslant 1}^{\mathrm{op}}} \hookrightarrow \mathbf{sSet}$ of Example 3.2.6 then embeds the category of graphs in the category of simplicial sets.

Alternatively, given a simplicial set $X$, we can take the 0-simplices as vertices, the non-degenerate 1-simplices as edges, and the face maps as the source and target maps to construct a graph. This construction fails to be functorial, however, since a simplicial map may send a non-degenerate to a degenerate simplex, whereas a map of graphs may not send an edge to a vertex. If we extended our notion of maps of graphs to account for that possibility, the resulting category of graphs would be equivalent to the category $\mathbf{Set}^{\mathbf{\Delta}_{\leqslant 1}^{\mathrm{op}}}$ of 1-truncated simplicial sets.

The above interpretation of graphs as (one-dimensional) simplicial sets also provides us with another way to think about simplicial sets, as higher-dimensional directed graphs: where the 1-simplices – we might call them '1-edges' – connect two 0-simplices (vertices) in a given order, we also have '2-edges' connecting three (not necessarily distinct) 1-edges in a certain order, and '3-edges' connecting four 2-edges, and so forth. ◊

**Definition 3.2.8** · The *standard n-simplex* $\Delta^n$ is the simplicial set

$$\Delta^n \coloneqq \mathrm{Hom}_{\mathbf{\Delta}}(-, \mathbf{n}) : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set},$$

that is, the contravariant functor out of $\mathbf{\Delta}$ which is represented by $\mathbf{n}$. The face and degeneracy maps of $\Delta^n$ act by precomposition of the coface and codegeneracy maps, respectively. The non-degenerate $k$-simplices of $\Delta^n$ are the injective maps $\mathbf{k} \to \mathbf{n}$. In particular, the only non-degenerate $n$-simplex of $\Delta^n$ is the identity map, and there are no non-degenerate $k$-simplices for $k > n$. ◊

This construction assembles into a covariant functor $\Delta^- : \mathbf{\Delta} \to \mathbf{sSet}$ where the components of the natural transformation $\Delta^- f : \Delta^m \to \Delta^n$ for $f : \mathbf{m} \to \mathbf{n}$ in $\mathbf{\Delta}$ are defined by postcomposition with $f$. Note that the functor $\Delta^-$ is precisely the Yoneda embedding $y : \mathbf{\Delta} \hookrightarrow \mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$. The following two lemmas are elementary consequences of the Yoneda lemma and this observation.

**Lemma 3.2.9** · *For a simplicial set $X$, there is an isomorphism $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) \cong X_n$ which is natural in $X$ and $n$. Under this isomorphism, a simplicial map $f : \Delta^n \to X$ corresponds to the $n$-simplex given by the value of $f_n : \mathrm{Hom}_{\mathbf{\Delta}}(\mathbf{n}, \mathbf{n}) \to X_n$ at the identity map $1_{\mathbf{n}} : \mathbf{n} \to \mathbf{n}$.*

Applying this lemma to the case where $X$ is a standard simplex $\Delta^m$, we see that $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^m, \Delta^n) \cong \mathrm{Hom}_{\mathbf{\Delta}}(\mathbf{m}, \mathbf{n})$.

**Lemma 3.2.10** (Yoneda embedding) · *The simplex category* $\mathbf{\Delta}$ *is fully embedded in* **sSet** *by* $\Delta^{-}$.

**Definition 3.2.11** · A *simplicial subset* of a simplicial set $X$ is a simplicial set $Y$ such that $Y_n \subseteq X_n$ for all $n$ and the face and degeneracy maps of $Y$ are restrictions of the corresponding face and degeneracy maps of $X$. ◊

*Example* 3.2.12 · The *boundary* $\partial\Delta^n$ *of the standard* $n$*-simplex* $\Delta^n$ is the simplicial subset of $\Delta^n$ whose $k$-simplices are $k$-simplices of $\Delta^n$ if $k < n$ and iterated degeneracies of $j$-simplices of $\Delta^n$ for $j < n$ if $k \geqslant n$. More precisely, the $k$-simplices for $k \geqslant n$ are in the image of a map $\Delta^n f : \Delta^n_j \to \Delta^n_k$ for a surjection $f : \mathbf{k} \to \mathbf{j}$ in $\mathbf{\Delta}$ with $j < n$. (It follows from the cosimplicial identities that these descriptions are equivalent, since the surjections in $\mathbf{\Delta}$ are exactly the maps that can be written as a composition of only coface maps.) ◊

*Remark* 3.2.13 · Since **sSet** is the category of functors from the small category $\mathbf{\Delta}^{\mathrm{op}}$ to the complete and cocomplete category of sets, the category of simplicial sets is also complete and cocomplete [21, Proposition 3.3.9]. The limits and colimits can be computed levelwise as limits and colimits in **Set**. For example, the product $X \times Y$ of simplicial sets $X$ and $Y$ consists of the sets $(X \times Y)_n = X_n \times Y_n$ and face and degeneracy maps

$$\begin{aligned} d_i &= (d_i^X, d_i^Y) : X_n \times Y_n \to X_{n-1} \times Y_{n-1}, \\ s_j &= (s_j^X, s_j^Y) : X_n \times Y_n \to X_{n+1} \times Y_{n+1}. \end{aligned}$$ ◊

## 3.3 Geometric realisation

In this subsection, we define a functor $|-| : \mathbf{sSet} \to \mathbf{Top}$ which builds a topological space, called the *geometric realisation*, from a simplicial set. Rather than defining it directly, we will define a simpler functor from the simplex category $\mathbf{\Delta}$ to **Top**, and show that we can extend it to the desired functor $|-|$.

**Definition 3.3.1** · The *category of simplices* of a simplicial set $X$ is the category of elements $\int X$ of the contravariant functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. Explicitly, the category of simplices of $X$ has as objects the simplices $(\mathbf{n} \in \mathbf{\Delta}, x \in X_n)$, and a map $(\mathbf{m}, x \in X_m) \to (\mathbf{n}, y \in X_n)$ is a map $f : \mathbf{m} \to \mathbf{n}$ in $\mathbf{\Delta}$ such that $Xf(y) = x$. Note that $\int X$ is small since $\mathbf{\Delta}$ is. We write $\Pi : \int X \to \mathbf{\Delta}$ for the forgetful functor $(\mathbf{n}, x \in X_n) \mapsto \mathbf{n}$. ◊

A simplicial map $f : X \to Y$ induces a functor $\int f : \int X \to \int Y$ of categories of simplices with $(\mathbf{n}, x \in X_n) \mapsto (n, f_n(x) \in Y_n)$ on objects and $g : (\mathbf{m}, x) \to (\mathbf{n}, y) \mapsto g : (\mathbf{m}, f_m(x)) \to (\mathbf{n}, f_n(y))$ on maps, which indeed preserves the chosen simplex by naturality of $f$. This makes $\int(-)$ into a functor $\mathbf{sSet} \to \mathbf{Cat}$.

The following categorical theorem, which says that the representable functors are 'dense' in the contravariant functor category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$, will be useful to define a functor out of $\mathbf{sSet} = \mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$.

**Theorem 3.3.2** (density [16, § III.7], [14, Theorem 6.2.17]) · *Let* $\mathcal{C}$ *be a small category and* $F : \mathcal{C}^{\mathrm{op}} \to$ **Set** *a functor. Then* $F$ *is the colimit of the diagram*

$$\int F \xrightarrow{\Pi} \mathcal{C} \xhookrightarrow{y} \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

*where* $\Pi : \int F \to \mathcal{C}$ *is the forgetful functor and* $y : \mathcal{C} \hookrightarrow \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ *is the Yoneda embedding.*

Note that we do not assume *a priori* that the functor category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ has colimits of shape $\int F$, although this category is actually complete and cocomplete if $\mathcal{C}$ is small (see Remark 3.2.13); the proof below also shows that $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ has colimits of the diagrams $y\Pi : \int F \to \mathbf{sSet}$.

*Proof.* We first construct a cone under $y\Pi$ with vertex $F$, that is, a natural transformation $\lambda : y\Pi \Rightarrow F$ such that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(-,A) \arrow[rr, Rightarrow, "f_*"] \arrow[dr, Rightarrow, "\lambda_{(A,a)}"'] & & \mathrm{Hom}_{\mathcal{C}}(-,B) \arrow[dl, Rightarrow, "\lambda_{(B,b)}"] \\ & F & \end{tikzcd}$$

commutes for every map $f : (A, a \in FA) \to (B, b \in FB)$ in $\int F$. By the Yoneda lemma, the component natural transformation $\lambda_{(C,c\in FC)} : \mathrm{Hom}_{\mathcal{C}}(-,C) \Rightarrow F$ corresponds to an element $x_{(C,c)} := \lambda_{(C,c),C}(1_C) \in FC$. Commutativity of the triangle above then is equivalent to

$$x_{(A,a)} = \lambda_{(A,a),A}(1_A) = \lambda_{(B,b),B}f_*(1_A) = Ff\lambda_{(A,a),A}(1_A) = Ffx_{(B,b)},$$

where the third equality follows from naturality of $\lambda$.

We can thus construct the desired cone by choosing elements $x_{(A,a)} \in FA$ for all $(A, a \in FA)$ in $\int F$ such that $Ffx_{(B,b)} = x_{(A,a)}$ for every map $f : (A, a \in FA) \to (B, b \in FB)$. For this, we can just take $x_{(A,a)} := a$, since being a map $f : (A,a) \to (B,b)$ in $\int F$ means that $Ffx_{(B,b)} = Ffb = a = x_{(A,a)}$.

We now show that the constructed cone is a colimiting cone. So let $G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be another functor with elements $y_{(A,a\in GA)} \in GA$ such that $Gfy_{(B,b)} = y_{(A,a)}$ for every map $f : (A, a \in GA) \to (B, b \in GB)$. We need to show that there is a unique natural transformation $\alpha : F \Rightarrow G$ such that $\alpha_A(x_{(A,a)}) = y_{(A,a)}$ for all $a \in FA$. But this requirement provides a unique definition for $\alpha$, since every element $a \in FA$ is of the form $a = x_{(A,a)}$. Naturality of $\alpha$ is easily verified: if $f : A \to B$ is a map in $\mathcal{C}$, then for all $b \in FB$:

$$\begin{tikzcd} FB \ni b = x_{(B,b)} \arrow[r, mapsto, "\alpha_B"] \arrow[d, "Ff"'] & y_{(B,b)} \in GB \arrow[d, "Gf"] \\ FA \ni x_{(A,a)} = a \arrow[r, mapsto, "\alpha_A"'] & y_{(A,a)} \in GA \end{tikzcd}$$

□

**Corollary** 3.3.3 · *A simplicial set $X$ is isomorphic to* colim $\Delta^-\Pi$*, where* $\Pi : \int X \to \mathbf{\Delta}$ *is the forgetful functor.*

We might also write the colimit as $\mathrm{colim}_{(\mathbf{n},x\in X_n)\in\int X} \Delta^n$ or even $\mathrm{colim}_{x\in X_n} \Delta^n$. The maps $\Delta^n \to X \cong \mathrm{colim}_{x\in X_n} \Delta^n$ are those corresponding to the $n$-simplices of $X$ by the isomorphism of Lemma 3.2.9.

**Definition** 3.3.4 · The *standard topological n-simplex* $|\Delta^n|$ is defined as the topological space

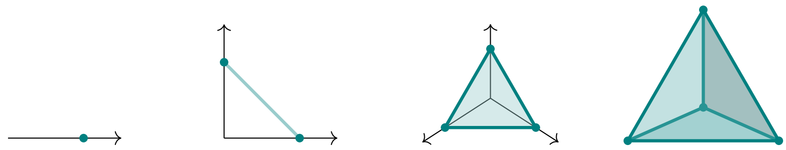
$$|\Delta^n| := \{\, (x_0,\ldots,x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1, x_i \geqslant 0 \,\}$$

with the subspace topology. ◊

The four standard topological simplices that can be embedded into three-dimensional space are projected onto the two-dimensional paper in Figure 3.1. These are respectively the point, the line segment, the triangle and the tetrahedron (all including their interior). Note that $|\Delta^n|$ is homeomorphic to the $n$-disk $D^n$ for all $n$.

A map $f : \mathbf{m} \to \mathbf{n}$ in $\mathbf{\Delta}$ induces a map $|\Delta^-|f : |\Delta^m| \to |\Delta^n|$ with $(x_0,\ldots,x_m) \mapsto (y_0,\ldots,y_n)$, where $y_i := \sum_{f(j)=i} x_j$. This defines a covariant functor $|\Delta^-| : \mathbf{\Delta} \to \mathbf{Top}$. By Lemma 3.2.10, we may see $|\Delta^-|$ as a functor from the full subcategory of **sSet** spanned by the representables $\Delta^n$ to the category of topological spaces.

*Remark* 3.3.5 · The notation $|\Delta^-|$ is suggestive, since we have already defined a functor $\Delta^-$ and are in the process of defining a functor $|-|$. We will justify this notation by making sure that the



**(i)** $|\Delta^0|$ **(ii)** $|\Delta^1|$ **(iii)** $|\Delta^2|$ **(iv)** $|\Delta^3|$

**Figure 3.1** · The first four standard topological simplices

following diagram commutes:

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, dashed, "|-|"] & \mathbf{Top} \\ \mathbf{\Delta} \arrow[u, hook, "\Delta^-"] \arrow[ur, "|\Delta^-|"'] & \end{tikzcd} \tag{3.2}$$

◊

**Definition 3.3.6** · The *geometric realisation* $|X|$ of a simplicial set $X$ is the topological space

$$|X| \coloneqq \operatorname{colim}_{\int X} |\Delta^-|\Pi = \operatorname{colim}_{x \in X_n} |\Delta^n|.$$

Being a colimit, there are maps $\chi_{(\mathbf{n}, x \in X_n)} : |\Delta^-|\Pi(\mathbf{n}, x \in X_n) = |\Delta^n| \to |X|$ for every $n$-simplex $x$ of $X$, the legs of the colimit cone, that assemble into a natural transformation, such that the diagram

$$\begin{tikzcd} |\Delta^{\Pi(\mathbf{m},x)}| = |\Delta^m| \arrow[rr, "|\Delta^-|f"] \arrow[dr, "\chi_{(\mathbf{m},x)}"'] & & |\Delta^n| = |\Delta^{\Pi(\mathbf{n},y)}| \arrow[dl, "\chi_{(\mathbf{n},y)}"] \\ & |X| & \end{tikzcd}$$

commutes for every map $f : (\mathbf{m}, x \in X_m) \to (\mathbf{n}, y \in X_n)$ in $\int X$. ◊

To show that $X \mapsto |X|$ on objects gives a functor $|-| : \mathbf{sSet} \to \mathbf{Top}$, we have to say what it does on maps. So let $f : X \to Y$ be a simplicial map, and write $\chi : |\Delta^-|\Pi \Rightarrow |X|$ and $\psi : |\Delta^-|\Pi \Rightarrow |Y|$ for the colimit cones of the geometric realisations of $X$ and $Y$. The whiskered composite $\psi \int f : |\Delta^-|\Pi \int f \Rightarrow |Y|$ is a natural transformation of functors $\int X \to \mathbf{Top}$. It is easy to see that the diagram of functors

$$\begin{tikzcd} \int X \arrow[rr, "\int f"] \arrow[dr, "\Pi"'] & & \int Y \arrow[dl, "\Pi"] \\ & \mathbf{\Delta} & \end{tikzcd}$$

commutes, so $\psi \int f$ is a cone under $|\Delta^-|\Pi : \int X \to \mathbf{Top}$. Applying the universal property of $\chi$ to this cone, we get a unique map $|f| : |X| \to |Y|$ that makes the following diagram commute for all $h : \mathbf{m} \to \mathbf{n}$:

$$\begin{tikzcd} |\Delta^m| \arrow[rr, "|\Delta^-|h"] \arrow[dr, "\chi"'] \arrow[ddr, bend right, "\psi\int f"'] & & |\Delta^n| \arrow[dl, "\chi"] \arrow[ddl, bend left, "\psi\int f"] \\ & |X| \arrow[d, dashed, "\exists! \, |f|"] & \\ & |Y| & \end{tikzcd}$$

This defines the functor $|-| : \mathbf{sSet} \to \mathbf{Top}$. That this construction is indeed functorial is straightforward to see: since $\int 1_X = 1_{\int X} : \int X \to \int X$, we have $|1_X| = 1_{|X|}$, and if $g : Y \to Z$ is a simplicial map and $\omega$ the colimit cone for $|Z|$, then $|g||f|$ and $|gf|$ both commute with $\omega \int g \int f = \omega \int gf$ and $\chi$, so they must be equal.

**Lemma** 3.3.7 · *The diagram* (3.2) *commutes: the realisation of the standard n-simplex is homeomorphic to the standard topological n-simplex.*

*Proof.* Since the category of simplices $\int\Delta^n$ of the standard $n$-simplex has a terminal object ($\mathbf{n} \in \mathbf{\Delta}, 1_{\mathbf{n}} \in \mathrm{Hom}_{\mathbf{\Delta}}(\mathbf{n}, \mathbf{n})$), the colimit of the functor $|\Delta^-|\Pi : \int\Delta^n \to \mathbf{Top}$ is the value of that functor at that terminal object. The realisation of $\Delta^n$ is thus homeomorphic to $|\Delta^n|$. □

*Remark* 3.3.8 · A more intuitive way to think about the geometric realisation of a simplicial set $X$, is as forming a disjoint union with a copy of $|\Delta^n|$ for each $n$-simplex of $X$, and then glueing the $i$th face of each topological standard $n$-simplex corresponding to $x \in X_n$ to the topological standard $(n-1)$-simplex corresponding to $d_i(x) \in X_{n-1}$. The degeneracy maps $s_j$ allow us to regard an $n$-simplex as an $(n+1)$-simplex. ◊

**Definition** 3.3.9 · If $Y$ is a simplicial subset of a simplicial set $X$, the *quotient simplicial set* $X/Y$ is the levelwise quotient of $X$ by $Y$, that is, the set $(X/Y)_n$ of $n$-simplices is the quotient set $X_n/Y_n$, where all elements of $Y_n$ are identified. Since $Y$ is a simplicial subset, the images of simplices of $Y$ under the face and degeneracy maps are also simplices of $Y$. Therefore, the obvious formulas for the face and degeneracy maps in $X/Y$ are well-defined with respect to the set quotients. The simplicial identities of $X/Y$ follow from those of $X$. ◊

*Example* 3.3.10 (simplicial spheres) · The quotient simplicial set $\Delta^1/\partial\Delta^1$ of the standard $n$-simplex by its boundary is called the *simplicial circle*, since its geometric realisation is homeomorphic to the circle $S^1$. This simplicial set has one 0-simplex •, the *point*, and one non-degenerate 1-simplex ∩, the *loop*, which connects the point • to itself. All higher simplices are degenerate. More generally, we can define the *simplicial n-sphere* as $\Delta^n/\partial\Delta^n$ for $n \geqslant 1$, the geometric realisation of which is the topological $n$-sphere $S^n$. The simplicial $n$-sphere is generated by a single 0-simplex and a single non-degenerate $n$-simplex. These 'simplicial models' for the topological spheres can be used to very directly compute the integral singular homology (Example 2.6.9) of the topological spheres. ◊

*Example* 3.3.11 · Via the interpretation of graphs as simplicial sets of Example 3.2.7, the realisation of the obtained simplicial sets provides a topological interpretation of graphs. The topological space corresponding in this way to a graph has points for the vertices of the graph, and line segments for the edges which connect the endpoints. ◊

## 3.4 Singular set

Since the geometric realisation $|-| : \mathbf{sSet} \to \mathbf{Top}$ is defined as a colimit, and colimits commute with colimits, the functor $|-|$ preserves them. It is thus plausible that the realisation functor has a right adjoint. In this subsection, we will indeed define a functor $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$, and show that it is indeed right adjoint to $|-|$.

**Definition** 3.4.1 · Given a topological space $X$, the *singular set* $\mathrm{Sing}\,X$ is the simplicial set with

$$(\mathrm{Sing}\,X)_n \coloneqq \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X).$$

This definition is seen to be functorial as the composition of the functors $|\Delta^-|^{\mathrm{op}} : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}^{\mathrm{op}}$ and $\mathrm{Hom}(-, X) : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$. On a map $h : \mathbf{m} \to \mathbf{n}$, the simplicial set $\mathrm{Sing}\,X$ is given by precomposition with the induced map $|\Delta^-|h : |\Delta^m| \to |\Delta^n|$.

The construction of the singular set defines a functor, the *singular functor* $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$. For a continuous map $f : X \to Y$, the components of the induced simplicial map $\mathrm{Sing}\,f : \mathrm{Sing}\,X \Rightarrow \mathrm{Sing}\,Y$ are defined by postcomposition with $f$. ◊

**Proposition** 3.4.2 · *The geometric realisation functor* $|-| : \mathbf{sSet} \to \mathbf{Top}$ *is left adjoint to the singular functor* $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$.

*Proof.* For a simplicial set $X$ and a topological space $Y$, we have the following string of isomorphisms natural in $X$ and $Y$:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) &= \mathrm{Hom}_{\mathbf{Top}}(\mathrm{colim}_{x\in X_n} |\Delta^n|, Y) \\
&\cong \lim_{x\in X_n} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) \\
&= \lim_{x\in X_n} (\mathrm{Sing}\, Y)_n \\
&\cong \lim_{x\in X_n} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{Sing}\, Y) && \text{(by Lemma 3.2.9)} \\
&\cong \mathrm{Hom}_{\mathbf{sSet}}(\mathrm{colim}_{x\in X_n} \Delta^n, \mathrm{Sing}\, Y) \\
&\cong \mathrm{Hom}_{\mathbf{sSet}}(X, \mathrm{Sing}\, Y) && \text{(by Corollary 3.3.3).}
\end{aligned}$$

The first and third isomorphisms follow from the categorical theorem [21, Theorem 3.4.7]. Note that the limits have shape $(\int X)^{\mathrm{op}}$, whereas the colimits are of shape $\int X$. □

*Remark* 3.4.3 ([21, Remark 6.5.9]) · The constructions of the realisation functor $|-| : \mathbf{sSet} \to \mathbf{Top}$ and its right adjoint $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ of the preceding sections provide a general strategy for constructing an adjunction of functors $L : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \rightleftarrows \mathcal{D} : R$ between the category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ of *presheaves* on some small category $\mathcal{C}$ and a cocomplete category $\mathcal{D}$:

(i) Define a functor $F : \mathcal{C} \to \mathcal{D}$.

(ii) Notice for a presheaf $G$ on $\mathcal{C}$ from the density theorem 3.3.2 that $G$ is isomorphic to $\mathrm{colim}_{x\in GX}\, yX$, where $y : \mathcal{C} \hookrightarrow \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the Yoneda embedding.

(iii) Define $L : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{D}$ by left Kan extension of $F$ along the Yoneda embedding $y$, explicitly $LG := \mathrm{colim}_{x\in GX}\, FX$, making the following diagram commute:

$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \arrow[r, dashed, "L"] & \mathcal{D} \\
\mathcal{C} \arrow[u, hook, "y"] \arrow[ur, "G"'] &
\end{tikzcd}$$

(iv) Define $RY := \mathrm{Hom}_{\mathcal{D}}(F-, Y) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, which is seen to be right adjoint to $L$. ◊

## 3.5 Model structure

**Definition 3.5.1** · For $n \geqslant 1$ and $0 \leqslant k \leqslant n$, the *$k$th horn* $\Lambda^n_k$ of $\Delta^n$ is the simplicial subset of the standard $n$-simplex $\Delta^n$ where the unique non-degenerate $n$-simplex and its $k$th face (its image under $d_k : \Delta^n_n \to \Delta^n_{n-1}$) are removed. More explicitly, the $m$-simplices of the horn $\Lambda^k_n$ coincide with those of $\Delta^n$ for $m < n-1$, the $(n-1)$-simplices are the $(n-1)$-simplices of $\Delta^n$ except for $d^k : \mathbf{n-1} \to \mathbf{n}$, and higher simplices are degenerate. The horn is also a simplicial subset of the boundary $\partial\Delta^n$. ◊

Hovey uses the entire chapter 3 of [13] to establish the following result.

**Theorem 3.5.2** ([13, Theorem 3.6.5, Proposition 3.2.2]) · *There is a cofibrantly generated model structure on the category* **sSet** *of simplicial sets with weak equivalences being maps $f$ such that the geometric realisation $|f|$ is a weak homotopy equivalence, the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geqslant 0$ as generating cofibrations and the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ for $n \geqslant 1$ and $0 \leqslant k \leqslant n$ as generating acyclic cofibrations. The cofibrations are precisely the levelwise injective simplicial maps, and every simplicial set is cofibrant.*

The weak equivalences in this model structure on **sSet** are thus created by the realisation functor $|-| : \mathbf{sSet} \to \mathbf{Top}$ with respect to the model structure of Theorem 2.5.3 on **Top**. The fibrations in this model category are called *Kan fibrations* and the fibrant objects are called *Kan complexes*.

The following important theorem shows that the homotopy categories of the model structures on the categories of topological spaces and simplicial sets are equivalent.

**Theorem 3.5.3** ([13, Theorem 3.6.7]) · *The adjunction* $|-| \dashv \mathrm{Sing}$ *of the geometric realisation and the singular functor between* **sSet** *and* **Top** *is a Quillen equivalence with respect to the model structures of Theorem 3.5.2 and Theorem 2.5.3.*

**Corollary 3.5.4** · *The homotopy categories* **Ho sSet** *and* **Ho Top** *with respect to the model structures of Theorem 3.5.2 and Theorem 2.5.3 are equivalent.*

*Proof.* Directly from Theorem 3.5.3 and Proposition 2.3.11. □

CHAPTER 4

# Group actions and symmetry

In this chapter, we turn to an *equivariant* setting, that is, a setting where the action of a group provides a form of symmetry. This chapter is primarily dedicated to definitions regarding group actions which we need in the next chapter, and does not contain many results itself. We present group actions in a categorical language in § 4.1. In § 4.2, we discuss the adjoints of the restriction functor from $G$-objects to $H$-objects for subgroups $H$ of $G$. The fixed-point and orbit functors, which are adjoint to the functor equipping a non-equivariant object with a trivial $G$-action, are discussed in § 4.3. Finally, in § 4.4, we briefly look at group actions on simplicial sets. The main source for this chapter is [21], which presents parts of these notions throughout several chapters.

*Convention* 4.0.1 · In the rest of this thesis, we will consider only *finite* groups. ◊

## 4.1 Group actions

Here we recall some definitions related to group actions and present them in a categorical language.

**Definition 4.1.1** · The *delooping groupoid* $\mathbf{B}G$ of a group $G$ is the category with a single object and a morphism at that object for each group element of $G$. Composition of morphisms is defined by multiplication of the corresponding group elements, and $\mathbf{B}G$ is indeed a groupoid since $G$ has inverses with respect to its multiplication. ◊

The mapping $G \mapsto \mathbf{B}G$ defines a functor $\mathbf{Grp} \to \mathbf{Grpd}$. A group homomorphism $f : H \to G$ thus induces a functor $\mathbf{B}f : \mathbf{B}H \to \mathbf{B}G$. Moreover, all functors $\mathbf{B}H \to \mathbf{B}G$ are of the form $\mathbf{B}f$ for some homomorphism $f$.

**Definition 4.1.2** · A (left) *group action* of a group $G$ on an object $X$ of a category $\mathcal{C}$ is a functor $\mathbf{B}G \to \mathcal{C}$ whose value at the unique object of $\mathbf{B}G$ is $X$. Equivalently, since maps in $\mathbf{B}G$ are automorphisms and functors preserve isomorphisms, a $G$-action is a group homomorphism $G \to \mathrm{Aut}_{\mathcal{C}} X$. A *$G$-object* in $\mathcal{C}$ is a pair of an object $X$ of $\mathcal{C}$ and a group action on $X$. The *category of $G$-objects in* $\mathcal{C}$ is the functor category $\mathcal{C}^{\mathbf{B}G}$. A map $f : X \to Y$ of $G$-objects is a natural transformation, and is called a *$G$-equivariant map* or simply a *$G$-map.* ◊

When $\mathcal{C}$ is a *concrete* category in the sense of [21, Definition 1.6.17], whose objects $X$ 'have points' (such as elements in **Set**, points in **Top**, vectors in $\mathbf{Vect}_k$, simplices in **sSet**, etc.), we will write $g \cdot x$ or $gx$ for the result of applying the action given by $g \in G$ to the point $x$ of $X$. More generally, we write $g_* : X \to X$ for the automorphism induced by a group element $g \in G$ on a $G$-object $X$. By definition of the $G$-object $X$, we have $e_* = 1_X$ and $(gh)_* = g_* h_*$ for all $g, h \in G$. Spelling out the definition using this notation, a map between $G$-objects $X$ and $Y$ is a map $f : X \to Y$ in $\mathcal{C}$ such that $g_* f = f g_*$ for all $g \in G$.

*Remark* 4.1.3 · Dually, a *right* $G$-object in a category $\mathcal{C}$ is a functor $\mathbf{B}G^{\mathrm{op}} \to \mathcal{C}$. Since $(\mathbf{B}G)^{\mathrm{op}} = \mathbf{B}(G^{\mathrm{op}})$, a right $G$-object is the same thing as a left $G^{\mathrm{op}}$-object. When dealing with right $G$-objects, we will write $g^*$ for the map induced by $g \in G$. In this case, we have $(gh)^* = h^* g^*$ for all $g, h \in G$. For concrete categories and right $G$-actions, we write the group elements on the right of 'points', as $x \cdot g$ or $xg$ for a point $x$. ◊

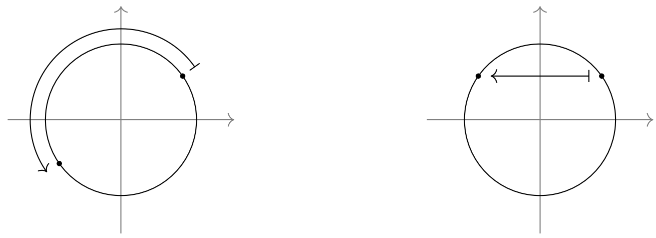
*Example* 4.1.4 · In the category of sets, $G$-objects are *$G$-sets.* A $G$-action on a set $X$ consists, for every group element, of a permutation of $X$ such that the action functor $\mathbf{B}G \to \mathbf{Set}$ takes multiplication of $G$ to composition of the corresponding permutations. ◊

*Example* 4.1.5 · In the category of topological spaces, a *$G$-space* is a space $X$ with a homeomorphism from $X$ to itself for every element of $G$, transferring the multiplication of $G$ into composition of the homeomorphisms. On the unit circle $S^1$ in $\mathbb{R}^2$, we can for example define the following two distinct actions of the cyclic group $C_2 = \langle\, g \mid g^2 = e\,\rangle$ of order two:

(i) $g$ acts by reflection through the origin $(x, y) \mapsto (-x, -y)$, or, equivalently, by rotation of $180^\circ$ around the origin;

(ii) $g$ acts by reflection through the $y$-axis $(x, y) \mapsto (-x, y)$.

These actions are visualised in Figure 4.1. ◊



**(i)** Rotation of $180^\circ$ around the origin **(ii)** Reflection through the $y$-axis

**Figure 4.1** · Two $C_2$-actions on the circle $S^1$

*Example* 4.1.6 · On the topological space (or vector space) $\mathbb{R}^n$, we can define an action of the symmetric group $S_n$, where a permutation $\sigma \in S_n$ of $\{1, \ldots, n\}$ acts by

$$\sigma_* : (x_1, \ldots, x_n) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \qquad ◊$$

*Example* 4.1.7 (trivial $G$-objects) · Any object $X$ in a category $\mathcal{C}$ can be equipped with the *trivial $G$-action*, where every group element $g \in G$ acts as the identity on $X$. This construction defines a functor $\operatorname{triv} : \mathcal{C} \to \mathcal{C}^{\mathbf{B}G}$ with $\operatorname{triv} f := f : \operatorname{triv} X \to \operatorname{triv} Y$ on maps $f : X \to Y$ in $\mathcal{C}$, which is clearly equivariant. ◊

## 4.2 Restriction and induction

Let $\mathcal{C}$ be a complete and cocomplete category. A group homomorphism $f : H \to G$ defines a functor $f^* := (\mathbf{B}f)^* : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}^{\mathbf{B}H}$ by precomposition with the induced functor $\mathbf{B}f : \mathbf{B}H \to \mathbf{B}G$. When $f$ is an inclusion $i : H \hookrightarrow G$ of a subgroup $H$ into its ambient group $G$, the functor $\operatorname{res}_H := (\mathbf{B}i)^* : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}^{\mathbf{B}H}$ restricts a $G$-object in $\mathcal{C}$ to an $H$-object, with the same underlying object of $\mathcal{C}$. In particular, the forgetful functor $U : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}$, given by $\operatorname{res}_e : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}^{\mathbf{B}e}$ followed by the isomorphism $\mathcal{C}^{\mathbf{B}e} \cong \mathcal{C}^{\mathbb{1}} \cong \mathcal{C}$, sends a $G$-object in $\mathcal{C}$ to the underlying object of $\mathcal{C}$ on which $G$ acts.

The restriction functor $(\mathbf{B}f)^* : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}^{\mathbf{B}H}$ along a homomorphism $f : H \to G$ has left and right adjoints given by Kan extensions along $\mathbf{B}f$ since it is defined by precomposition [21, Corollary 6.2.6]. For subgroup inclusions $i : H \hookrightarrow G$, the adjoints of $\operatorname{res}_H = (\mathbf{B}i)^*$ are called *induction* $\mathrm{ind}_H : \mathcal{C}^{\mathbf{B}H} \to \mathcal{C}^{\mathbf{B}G}$ and *coinduction* $\mathrm{coind}_H : \mathcal{C}^{\mathbf{B}H} \to \mathcal{C}^{\mathbf{B}G}$:

$$\begin{tikzcd} \mathcal{C}^{\mathbf{B}G} \arrow[r, "\mathrm{res}_H" description] & \mathcal{C}^{\mathbf{B}H} \arrow[l, bend right, "\mathrm{ind}_H"', "\perp"] \arrow[l, bend left, "\mathrm{coind}_H", "\perp"'] \end{tikzcd} \tag{4.1}$$

By computing the Kan extensions, we obtain the following explicit description of the induction functor, see [21, Example 6.2.8]. For $X : \mathbf{B}H \to \mathcal{C}$, the underlying object of $\mathrm{ind}_H X$ is isomorphic to $\coprod_{G/H} X$. Following [23] (which we discuss in § 5.2), we write $G/H \otimes X \coloneqq \coprod_{G/H} X$ for this object. Let $G/H = \{ g_i H \mid g_i \in G \}$ be a complete set of representatives of left $H$-cosets; a group element $g \in G$ factors uniquely as $g = g_i h$ for $g_i$ such a representative and $h \in H$. The $G$-action on $\mathrm{ind}_H X \cong G/H \otimes X$ is then given by $g_* \circ \mathrm{inj}_{g_i H} \coloneqq \mathrm{inj}_{g_j H} \circ h'_*$ if $g g_i = g_j h'$ with $h' \in H$, where we write $\mathrm{inj}_{g_i H} : X \to G/H \otimes X$ for the coproduct inclusion at $g_i H$. In particular, when $H$ is the trivial subgroup, so that $\mathrm{ind}_e$ is the left adjoint of the forgetful functor $U : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}$, we see $\mathrm{ind}_e X \cong G/e \otimes X \cong G \otimes X$ with $G$ acting by left multiplication on the indexing set; this is the *free* $G$-object on $X$.

The description of $\mathrm{coind}_H$ is dual; as objects of $\mathcal{C}$, we have $\mathrm{coind}_H X \cong \prod_{H\backslash G} X =: H\backslash G \pitchfork X$, where $H\backslash G$ is the set of right $H$-cosets. The right adjoint of the forgetful functor $U : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}$ is $G \pitchfork - = \prod_G(-)$ with $G$ acting on the right of the indexing set.

*Example* 4.2.1 · The *cofree* functor $C_n \pitchfork -$ applied to the topological space $\mathbb{R}$ induces an action of the cyclic group $C_n = \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{R}^n$. An element $i \in \mathbb{Z}/n\mathbb{Z}$ acts on this space by

$$i_* : (x_1, \ldots, x_n) \mapsto (x_{1+i}, \ldots, x_{n+i}),$$

where addition happens modulo $n$. Dually, applying the free functor $C_n \otimes -$ to $\mathbb{R}$, we obtain a $C_n$-action on $\coprod_{i=1}^n \mathbb{R}$ where $i_*$ sends $\mathrm{inj}_j(x)$ to $\mathrm{inj}_{i+j}(x)$ with addition modulo $n$. ◊

## 4.3 Fixed points and orbits

Let $\mathcal{C}$ be a complete and cocomplete category. Recognising the trivial functor from Example 4.1.7 as the constant diagram functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathbf{B}G}$, the left and right adjoints are respectively the colimit and limit functor:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{triv}" description] & \mathcal{C}^{\mathbf{B}G} \arrow[l, bend right, "\mathrm{colim}"', "\perp"] \arrow[l, bend left, "\lim", "\perp"'] \end{tikzcd} \tag{4.2}$$

The limit of a $G$-object $X$ in $\mathcal{C}$ (that is, a diagram in $\mathcal{C}$ of shape $\mathbf{B}G$) is an object $\lim X$ of $\mathcal{C}$ with a map $\lim X \to X$ such that the diagram

$$\lim X \longrightarrow X \underset{h_*}{\overset{g_*}{\rightrightarrows}} X$$

commutes for all $g, h \in G$. Dually, the colimit of $X$ is an object $\mathrm{colim}\, X$ with a map $X \to \mathrm{colim}\, X$ such that the diagram

$$X \underset{h_*}{\overset{g_*}{\rightrightarrows}} X \longrightarrow \mathrm{colim}\, X$$

commutes for all $g, h \in G$.

*Example* 4.3.1 · In the category of sets, the limit of a $G$-set $X$ is a set $Y$ with a map $i : Y \to X$ such that $g \cdot i(x) = h \cdot i(x)$ for all $g, h \in G$ and $x \in X$. Thus, $Y$ is (isomorphic to) the set of $G$-*fixed points* of $X$:

$$X^G := \{ x \in X \mid gx = x \text{ for all } g \in G \}.$$

Dually, the colimit of $X$ is a set $Z$ with a map $q : X \to Z$ such that $q(gx) = q(hx)$ for all $g, h \in G$ and $x \in X$. The set $Z$ is (isomorphic to) the set of $G$-*orbits* of $X$:

$$X_G := \{ Gx \mid x \in X \}, \quad \text{where } Gx := \{ gx \mid g \in G \}.$$

The sets $Gx$ are seen to be the equivalence classes of the $G$-orbit relation $\sim_G$ generated by $x \sim_G gx$ for all $x \in X$ and $g \in G$; the set $X_G$ is then the quotient set $X/{\sim_G}$.

Since there is often a strong connection between limits and colimits in the categories we consider and those in the category of sets, this example is helpful to describe the (co)limits of shape $\mathbf{B}G$ in those categories. For instance, in the category of topological spaces, (co)limits are computed by topologising the respective (co)limits in **Set**; this follows from the fact that the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ has both left and right adjoints (given by equipping a set with the discrete or indiscrete topology, respectively), and hence preserves (co)limits. The underlying sets of the fixed-point space and orbit space of a $G$-space are thus the sets of fixed points and orbits. ◊

**Definition 4.3.2** · The *fixed-point object* $X^G$ of a $G$-object $X$ in a category $\mathcal{C}$ is, if it exists, the limit of the diagram $X : \mathbf{B}G \to \mathcal{C}$. Dually, the *orbit object* $X_G$ of $X$ is, if it exists, the colimit of the diagram $X : \mathbf{B}G \to \mathcal{C}$. The *fixed-point functor* $(-)^G : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}$ and *orbit functor* $(-)_G : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}$ are respectively the limit and colimit functors.

For a subgroup $H$ of $G$, we define the $H$-*fixed-point object* $X^H$ and the $H$-*orbit object* $X_H$ of a $G$-object $X$ in a category $\mathcal{C}$ as respectively the limit and colimit of the restricted diagram $\mathrm{res}_H X : \mathbf{B}H \to \mathcal{C}$. The $H$-*fixed-point functor* $(-)^H : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}$ and $H$-*orbit functor* $(-)_H : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}$ are defined as the composite of the restriction functor $\mathrm{res}_H : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}^{\mathbf{B}H}$ and the limit functor $\lim_{\mathbf{B}H} : \mathcal{C}^{\mathbf{B}H} \to \mathcal{C}$ or colimit functor $\mathrm{colim}_{\mathbf{B}H} : \mathcal{C}^{\mathbf{B}H} \to \mathcal{C}$, respectively. ◊

The orbit object $X_H$ of a $G$-object $X$ is also often denoted $X/H$. To emphasise the duality of orbit and fixed-point objects, which will be central in Chapter 5, we use the notation $X_H$ throughout.

Since the $H$-fixed-point functor $(-)^H : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}$ is the composite of the restriction functor $\mathrm{res}_H : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}^{\mathbf{B}H}$ which has a left adjoint $\mathrm{ind}_H = G/H \otimes -$ and the limit functor $\lim_{\mathbf{B}H} : \mathcal{C}^{\mathbf{B}H} \to \mathcal{C}$ which has a left adjoint triv, the composite of these left adjoints is left adjoint to $(-)^H$:

$$\begin{tikzcd}
\mathcal{C}^{\mathbf{B}G} \arrow[rr, "(-)^H"] \arrow[dr, shift left, "\mathrm{res}_H"] & & \mathcal{C} \arrow[dl, shift left, "\mathrm{triv}"] \\
& \mathcal{C}^{\mathbf{B}H} \arrow[ul, shift left, "\mathrm{ind}_H"] \arrow[ur, shift left, "\lim_{\mathbf{B}H}"] &
\end{tikzcd}$$

This composite sends an object $X$ of $\mathcal{C}$ to the $G$-object $G/H \otimes X$ with the canonical left $G$-action on the indexing set $G/H$ and the trivial action on the factors $X$. By slight abuse of notation, we also write $G/H \otimes - : \mathcal{C} \to \mathcal{C}^{\mathbf{B}G}$ for this functor. Dually, the right adjoint of the $H$-orbit functor $(-)_H : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}$ is $H \backslash G \pitchfork -$.

*Example* 4.3.3 · The category **Top** of topological spaces is complete and cocomplete, so fixed-point and orbit objects exist for any $G$-space $X$. The fixed-point space $X^G \hookrightarrow X$ has the fixed-point set $X^G$ of Example 4.3.1 as its set of points with the subspace topology. Dually, the orbit space $X \twoheadrightarrow X_G$ is the quotient space $X/{\sim_G}$ of $X$ by the $G$-orbit relation $\sim_G$. ◊

*Example* 4.3.4 · For the $C_2$-actions on the circle $S^1$ of Example 4.1.5:

(i) No point on the circle is fixed by a rotation of 180° around the origin, so the fixed-point space of this action is the empty space. The orbits of the action are the sets $\{x, -x\}$ of antipodal points on the circle. Forming the quotient $S^1/(x \sim -x)$ where the antipodes are identified gives a circle again, so the orbit space is homeomorphic to $S^1$.

(ii) Reflection through the $y$-axis only fixes the 'north' and 'south pole' of the circle, the points $(0,1)$ and $(0,-1)$. The fixed-point space of this action is thus a disjoint union of two copies of the one-point space. The orbit space is seen to be homeomorphic to the unit interval $[0,1]$.

From the functoriality of the fixed-point functor $(-)^{C_2} : \mathbf{Top}^{\mathrm{B}C_2} \to \mathbf{Top}$, it follows that there is no $C_2$-equivariant map from the latter to the former space, since such a $C_2$-map would induce a map $\{(0,1),(0,-1)\} \to \emptyset$ on fixed-point spaces. ◊

## 4.4 Group actions on simplicial sets

Recall from Remark 3.2.2 that $\mathbf{s}\mathcal{C} = \mathcal{C}^{\Delta^{\mathrm{op}}}$ is the category of simplicial objects of a category $\mathcal{C}$.

**Lemma 4.4.1** · *The categories* $\mathbf{sSet}^{\mathrm{B}G}$ *and* $\mathbf{s}(\mathbf{Set}^{\mathrm{B}G})$ *are isomorphic.*

*Proof.* Writing $[\mathcal{C},\mathcal{D}]$ for the functor category $\mathcal{C}^{\mathcal{D}}$ and using the currying isomorphism $[\mathcal{C}\times\mathcal{D},\mathcal{E}] \cong [\mathcal{C},[\mathcal{D},\mathcal{E}]]$, we see:

$$\begin{aligned}\mathbf{sSet}^{\mathrm{B}G} &= [\mathrm{B}G, \mathbf{sSet}]\\ &= [\mathrm{B}G, [\Delta^{\mathrm{op}}, \mathbf{Set}]]\\ &\cong [\mathrm{B}G\times\Delta^{\mathrm{op}}, \mathbf{Set}]\\ &\cong [\Delta^{\mathrm{op}}\times\mathrm{B}G, \mathbf{Set}]\\ &\cong [\Delta^{\mathrm{op}}, [\mathrm{B}G, \mathbf{Set}]]\\ &= \mathbf{s}[\mathrm{B}G,\mathbf{Set}] = \mathbf{s}(\mathbf{Set}^{\mathrm{B}G}).\end{aligned}$$ □

In other words, a $G$-simplicial set is the same object as a simplicial $G$-set. A $G$-simplicial set thus consists of a sequence of $G$-sets with equivariant face and degeneracy maps.

Since the category of simplicial sets is complete and cocomplete (Remark 3.2.13), the left and right adjoints of the restriction functors of (4.1) and of the trivial functor of (4.2) exist. Because limits and colimits in **sSet** can be computed levelwise as limits and colimits in **Set**, the fixed-point simplicial set of a $G$-simplicial set $X$ has as its $n$-simplices the fixed points of the $G$-set $X_n$. Dually, the $n$-simplices of the orbit simplicial set of $X$ are the orbits of the $G$-set $X_n$.

*Example* 4.4.2 · Define a $G$-action on the nerve $NG$ of $G$ (Example 3.2.5) by conjugation on all group elements in the simplices: an element $g \in G$ acts by

$$g_* : (x_1,\ldots,x_n) \mapsto (gx_1g^{-1},\ldots,gx_ng^{-1}).$$

It is easy to see that the face and degeneracy maps of $NG$ are equivariant with respect to these actions on the simplices; for the face map $d_i : NG_n \to NG_{n-1}$ with $0 < i < n$, for example, we see:

$$\begin{tikzcd}(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) \arrow[r, mapsto, "g_*"] \arrow[d, mapsto, "d_i"'] & (gx_1g^{-1},\ldots,gx_ig^{-1},gx_{i+1}g^{-1},\ldots,gx_ng^{-1}) \arrow[d, mapsto, "d_i"] \\ (x_1,\ldots,x_ix_{i+1},\ldots,x_n) \arrow[r, mapsto, "g_*"] & (gx_1g^{-1},\ldots,gx_ix_{i+1}g^{-1},\ldots,gx_ng^{-1})\end{tikzcd}$$

The fixed-point group of the action of $G$ on itself by conjugation is the subgroup $\mathrm{Z}G$, the centre of $G$. The fixed simplices are thus exactly those sequences that contain only elements of $\mathrm{Z}G$. The fixed-point simplicial set of the $G$-action on $NG$ by conjugation, hence, is the nerve $N\mathrm{Z}G$ of the centre of $G$. If $G$ is abelian, then $\mathrm{Z}G = G$ and $NG$ is its own fixed-point simplicial set.

The orbits of the action of $G$ on itself by conjugation are the conjugacy classes of $G$, so the orbit of an $n$-simplex $(x_1,\ldots,x_n)$ consists of sequences $(y_1,\ldots,y_n)$ such that $x_i$ and $y_i$ are conjugate for all $i$, and $y_i = y_j$ whenever $x_i = x_j$. ◊

CHAPTER 5

# Equivariant homotopy theory

In this chapter, our goal is to define a model structure on the category $\mathcal{C}^{\mathbf{B}G}$ of $G$-objects in a model category $\mathcal{C}$ for a finite group $G$, specifically where $\mathcal{C}$ is the category of topological spaces or simplicial sets. In § 5.2, we describe the traditional way to do this, by taking a $G$-map $f : X \to Y$ in $\mathcal{C}$ to be a weak equivalence or fibration when the induced map $f^H : X^H \to Y^H$ on fixed points is respectively a weak equivalence or fibration for subgroups $H$ of $G$. The corresponding model structure is right induced from the projective model structure on the category of contravariant orbit diagrams in $\mathcal{C}$, whose domain category is presented in § 5.1. Elmendorf's theorem 5.2.6 shows that the adjunction between the categories of $G$-objects and contravariant orbit diagrams is a Quillen equivalence. The main sources we use here are [3], [23].

In § 5.3, following [7], we explore the dual problem: when is it possible to use the orbit functors $(-)_H : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}$ for all subgroups $H$ of $G$ to lift a model structure from $\mathcal{C}$ to $\mathcal{C}^{\mathbf{B}G}$? We will see that this is possible when $\mathcal{C}$ is the category of simplicial sets, but that the theory does not apply to the category of topological spaces. The duality between the model structures 'via fixed points' and 'via orbits' fails when we consider Elmendorf's theorem, and we present a counterexample to its dual version. Finally, we discuss an application of the model structure via orbits: a model categorical criterion for when maps that induce weak equivalences on orbits also induce weak equivalences on fixed points.

We recall that the groups we consider are assumed to be finite (Convention 4.0.1).

## 5.1 Orbit category

**Definition 5.1.1** · A *family of subgroups* $\mathcal{F}$ of a group $G$ is a non-empty collection of subgroups of $G$ which is closed under conjugation and taking subgroups. Explicitly: if $H \in \mathcal{F}$, then $g^{-1}Hg \in \mathcal{F}$ for all $g \in G$ and $K \in \mathcal{F}$ for all subgroups $K$ of $H$. ◊

The requirement that a family be non-empty, or equivalently that it contains the trivial subgroup, is not standard in the literature. We will, however, not be interested in empty families here.

*Example* 5.1.2 · For a group $G$, examples of families of subgroups of $G$ are:

(i) the set of all subgroups of $G$;

(ii) the set of only the trivial subgroup, which is the smallest family of subgroups of any group;

(iii) for a chosen subgroup $H$ of $G$, the set of all conjugates $g^{-1}Hg$ for $g \in G$ and their subgroups. ◊

**Definition 5.1.3** · For a family of subgroups $\mathcal{F}$ of a group $G$, the *orbit category* $\mathbf{Orb}_{\mathcal{F}}$ is the full subcategory of $\mathbf{Set}^{\mathbf{B}G}$ on the $G$-sets $G/H$ of left $H$-cosets with the $G$-action of left multiplication for all $H \in \mathcal{F}$. We write $\mathbf{Orb}_G$ for $\mathbf{Orb}_{\mathcal{F}}$ when $\mathcal{F}$ contains all subgroups of $G$. ◊

*Remark* 5.1.4 · A more general definition of the orbit category in the case that $G$ is a topological group is prevalent in the literature. Here, we will only be treating finite groups, however. The definition given above is identical to the original in [4, § 1.3] for the family of all subgroups. ◊

Let $\mathcal{F}$ be a family of subgroups of a group $G$. A map $G/H \to G/K$ in $\mathbf{Orb}_{\mathcal{F}}$ is determined by the image of the unit coset $eH$; for the map $G/H \to G/K$ with $eH \mapsto gK$, we write $\widetilde{g} : G/H \to G/K$ We have $\widetilde{g} : xH \mapsto xgK$, so $\widetilde{g} \circ \widetilde{h} = \widetilde{hg}$ for all $g, h \in G$ when $\widetilde{g}$ and $\widetilde{h}$ are well-defined and composable. When $\widetilde{g}$ and $\widetilde{h}$ are parallel maps $G/H \to G/K$, they are equal if and only if $gK = hK$, that is, $g^{-1}h \in K$. The elements $g \in G$ for which the map $\widetilde{g} : G/H \to G/K$ exists are exactly those such that $gK$ is an $H$-fixed point of $G/K$ with the canonical left $G$-action.

**Lemma 5.1.5** · *For $g \in G$, the assignment $eH \mapsto gK$ defines a map $\widetilde{g} : G/H \to G/K$ in $\mathbf{Orb}_{\mathcal{F}}$ if and only if $gK \in (G/K)^H$. In particular, there is an isomorphism $\mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}}(G/H, G/K) \cong (G/K)^H$ of sets.*

*Proof.* The map $\widetilde{g} : G/H \to G/K$ with $\widetilde{g}(eH) = gK$ is well-defined if and only if $\widetilde{g}(hH) = gK$ for all $h \in H$. This means that $gK = \widetilde{g}(eH) = \widetilde{g}(hH) = h \cdot \widetilde{g}(eH) = h(gK)$ for all $h \in H$, exactly expressing that $gK$ is an $H$-fixed point. □

Alternatively, the maps $G/H \to G/K$ correspond to subconjugation relations $g^{-1}Hg \subseteq K$.

**Lemma 5.1.6** · *The assignment $eH \mapsto gK$ defines a map $\widetilde{g} : G/H \to G/K$ in $\mathbf{Orb}_{\mathcal{F}}$ if and only if $g^{-1}Hg \subseteq K$.*

*Proof.* The subconjugation relation $g^{-1}Hg \subseteq K$ holds if and only if $g^{-1}hg \in K$ for all $h \in H$, which is equivalent to $h(gK) = gK$ for all $h \in H$. Lemma 5.1.5 then implies the result. □

It follows that the map $\widetilde{g} : G/H \to G/K$ in $\mathbf{Orb}_{\mathcal{F}}$ factors as an isomorphism $\widetilde{g} : G/H \to G/g^{-1}Hg$ followed by the map $\widetilde{e} : G/g^{-1}Hg \to G/K$. In particular, we have the following result about the endomorphisms, where

$$\mathrm{N}_G H = \{\, g \in G \mid g^{-1}Hg = H \,\}$$

is the *normaliser* of $H$ in $G$, of which $H$ is a normal subgroup by definition.

**Lemma 5.1.7** · *The monoid $\mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}}(G/H, G/H)$ of endomorphisms of $G/H$ in $\mathbf{Orb}_{\mathcal{F}}$ is isomorphic to $\mathrm{N}_G H/H$. In particular, every endomorphism in $\mathbf{Orb}_{\mathcal{F}}$ is an automorphism.*

For the argument below, it is necessary that $G$ is finite.

*Proof.* By the previous lemma, sending $g \in \mathrm{N}_G H$ to the endomorphism $\widetilde{g^{-1}} : eH \mapsto g^{-1}H$ (since $\widetilde{h} \circ \widetilde{k} = \widetilde{kh}$) of $G/H$ defines a map $\varphi : \mathrm{N}_G H \to \mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}}(G/H, G/H)$ of monoids. This map $\varphi$ is surjective, since if $\widetilde{g^{-1}} : G/H \to G/H$ is a map in $\mathbf{Orb}_{\mathcal{F}}$, then $gHg^{-1} \subseteq H$ by Lemma 5.1.6, which implies equality $gHg^{-1} = H$ since $G$ is finite. It follows that $g^{-1} \in \mathrm{N}_G H$ and thus $g \in \mathrm{N}_G H$. An endomorphism $\varphi(g)$ in the image of $\varphi$ has an inverse given by $\varphi(g^{-1})$, so we see that endomorphisms of $G/H$ are automorphisms and $\mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}}(G/H, G/H)$ is a group. The kernel of $\varphi$ is precisely $H$, whence the first isomorphism theorem of groups gives the desired result. □

## 5.2 Via fixed points

The traditional way to do equivariant homotopy theory, that is, homotopy theory of spaces with a group action, is to lift a model structure from the non-equivariant setting using fixed-point functors $(-)^H : \mathcal{C}^{\mathbf{B}G} \to \mathcal{C}$ for all subgroups $H$ in a family $\mathcal{F}$ of subgroups of $G$. This model structure is right induced from the projective model structure on the category of contravariant orbit diagrams. An important result is Elmendorf's theorem 5.2.6, which establishes a Quillen equivalence between these model structures. In this section, we discuss the model structure on $\mathcal{C}^{\mathbf{B}G}$ via fixed points for cofibrantly generated model categories $\mathcal{C}$ and prove Elmendorf's theorem, following [23].

From Lemma 5.1.7, it follows that the endomorphism group of $G/e$ in $\mathbf{Orb}_{\mathcal{F}}$ is isomorphic to $G$, and hence that $\mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}}(G/e, G/e) \cong G^{\mathrm{op}} \cong G$, with $g \in G$ corresponding to $\widetilde{g} : G/e \to G/e$. We can thus see $G/e$ as a left $G$-object in $\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}$, which defines an embedding $i : \mathbf{B}G \hookrightarrow \mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}$ sending the unique object of $\mathbf{B}G$ to $G/e$ and $g \in G$ to $\widetilde{g}$. Writing $[\mathcal{C}, \mathcal{D}]$ for the functor category $\mathcal{D}^{\mathcal{C}}$, precomposition with $i$ now defines a functor $\mathrm{ev}_{G/e} := i^* : [\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}, \mathcal{C}] \to [\mathbf{B}G, \mathcal{C}]$ which restricts a contravariant orbit diagram $\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}} \to \mathcal{C}$ to the full subcategory on the single object $G/e$. If $\mathcal{C}$ has certain limits, then $\mathrm{ev}_{G/e}$ has a right adjoint given by right Kan extension along $i$.

Assuming the fixed-point functors $(-)^H : [\mathbf{B}G, \mathcal{C}] \to \mathcal{C}$ of Definition 4.3.2 exist for all $H \in \mathcal{F}$ (the 'certain limits' from above), we can define this right adjoint $\Phi : [\mathbf{B}G, \mathcal{C}] \to [\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}, \mathcal{C}]$ by currying the functor $[\mathbf{B}G, \mathcal{C}] \times \mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}} \to \mathcal{C}$ which sends a pair $(X, G/H)$ to the fixed-point object $X^H$ and a pair of maps $f : X \to Y$ and $\widetilde{g} : G/H \to G/K$ to the unique map $X^K \to Y^H$, whose existence follows from Lemma 5.1.6, the equivariance of $f$ and the universal property of the fixed-point object $Y^H$, that makes the diagram

$$\begin{tikzcd}
X^K \arrow[r, "i_K^X"] \arrow[d, dashed] & X \arrow[r, "g_*"] & X \arrow[d, "f"] & \\
Y^H \arrow[rr, "i_H^Y"'] & & Y \arrow[r, shift left, "h_*"] \arrow[r, shift right, "h'_*"'] & Y
\end{tikzcd}$$

commute for all $h, h' \in H$, where $i_K^X$ and $i_H^X$ are the fixed-point inclusion maps. In particular, if $\widetilde{g}$ is the identity on $G/H$, then $f$ is sent to the map $f^H : X^H \to Y^H$ induced by the fixed-point functor $(-)^H : [\mathbf{B}G, \mathcal{C}] \to \mathcal{C}$.

By showing that $\Phi$ is given by right Kan extension along $i$, we see that $\Phi$ is indeed right adjoint to $\mathrm{ev}_{G/e}$. (Conversely, we could also find the definition of $\Phi$ by computing the right Kan extension, as is done in [21, Example 6.2.11].)

**Lemma 5.2.1** · *The functor* $\mathrm{ev}_{G/e} : [\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}, \mathcal{C}] \to [\mathbf{B}G, \mathcal{C}]$ *restricting to* $G/e$ *is left adjoint to the functor* $\Phi : [\mathbf{B}G, \mathcal{C}] \to [\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}, \mathcal{C}]$.

*Proof.* Let $X$ be a $G$-object. We show that $\Phi X$ is a right Kan extension of $X$ along $i : \mathbf{B}G \hookrightarrow \mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}$, which implies that $\Phi$ is right adjoint to $\mathrm{ev}_{G/e}$. A natural transformation $\alpha : \Phi X \circ i \Rightarrow X$ is just a $G$-map $\alpha : X^e \to X$, for which we can take the fixed-point inclusion map $X^e \to X$, which is an isomorphism. For any functor $F : \mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}} \to \mathcal{C}$, a $G$-map $\beta : F(G/e) \to X$ clearly factors uniquely through the isomorphism $\alpha$, showing that $\Phi X$ is a right Kan extension of $X$ along $i$. $\square$

The following assumptions, which are satisfied by the categories of topological spaces and simplicial sets, are used to show the existence of the model structure lifted using fixed-point functors.

**Definition 5.2.2** ([23, Proposition 2.6]) · Let $\mathcal{C}$ be a model category, $\mathcal{F}$ a family of subgroups of $G$, and $H \in \mathcal{F}$. We say that the $H$-fixed-point functor $(-)^H : [\mathbf{B}G, \mathcal{C}] \to \mathcal{C}$ satisfies the *cellularity conditions* if:

(i) $(-)^H$ preserves directed colimits of diagrams in which each underlying map is a cofibration;

(ii) $(-)^H$ preserves pushouts of maps of the form

$$G/K \otimes f : G/K \otimes A \to G/K \otimes B$$

for $K \in \mathcal{F}$ and $f : A \to B$ a cofibration in $\mathcal{C}$; and

(iii) for all $K \in \mathcal{F}$ and for each object $A$ of $\mathcal{C}$, the map $(G/K)^H \otimes A \to (G/K \otimes A)^H$ induced by the universal property of the fixed-point object from the coproduct inclusion $(G/K)^H \otimes A \hookrightarrow G/K \otimes A$ along $(G/K)^H \hookrightarrow G/K$ is an isomorphism. $\lozenge$

**Proposition 5.2.3** ([23, Proposition 2.6]) · *Let $\mathcal{C}$ be a cofibrantly generated model category and $\mathcal{F}$ a family of subgroups of $G$. If for all $H \in \mathcal{F}$ the $H$-fixed-point functors satisfy the cellularity conditions, then there is a cofibrantly generated $\mathcal{F}$-*model structure *on the category $[\mathbf{B}G, \mathcal{C}]$ of left $G$-objects where weak equivalences and fibrations are created by all fixed-point functors $(-)^H : [\mathbf{B}G, \mathcal{C}] \to \mathcal{C}$ for $H \in \mathcal{F}$.*

Using the terminology of [11, Definition 2.1.3], the $\mathcal{F}$-model structure is *right-induced* from the projective model structure on the category $[\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}, \mathcal{C}]$ of contravariant $\mathcal{F}$-orbit diagrams.

The motivating example of a model category for which the fixed-point functors satisfy the cellularity conditions, and which thus admits the $\mathcal{F}$-model structure, is the category of topological spaces with the model structure of Theorem 2.5.3 [23, Lemma 3.18]; this model structure is the classical approach to equivariant homotopy theory [3, Chapter 1]. For the category of simplicial sets with the model structure of Theorem 3.5.2, the fixed-point functors also satisfy the cellularity conditions, as the following lemma shows. The category of $G$-simplicial sets thus also admits the $\mathcal{F}$-model structure.

**Lemma 5.2.4** · *The cellularity conditions of Definition 5.2.2 hold for the category of simplicial sets with the model structure of Theorem 3.5.2.*

*Proof.* Since the cofibrations in **sSet** are levelwise injective maps and (co)limits in **sSet** are computed levelwise, it suffices to check the cellularity conditions in **Set** for injective maps.

For condition (i), let $X : \mathcal{J} \to [\mathbf{B}G, \mathbf{Set}]$ be a directed diagram of injective equivariant maps. We may assume that $Xf : X_j \hookrightarrow X_k$ is an inclusion of sets if there is a map $f : j \to k$ in $\mathcal{J}$; the $G$-action on $X_k$ then extends the action on $X_j$ by equivariance of $Xf$. The colimit of $X$ is the union $\operatorname{colim} X = \bigcup_{j\in\mathcal{J}} X_j$ with the $G$-action also given by extension of the actions of the $X_j$. Fixed-points sets $X_j^H$ are simply subsets $X_j^H \subseteq X_j$, so it follows that the colimit of $X^H = (-)^H \circ X$ is the union $\operatorname{colim} X^H = \bigcup_{j\in\mathcal{J}} X_j^H$, which is clearly a subset of $\operatorname{colim} X$.

For condition (ii), let $i : A \hookrightarrow B$ be an inclusion in the left-hand pushout square in $[\mathbf{B}G, \mathbf{Set}]$:

$$\begin{tikzcd}
G/K \otimes A \arrow[r] \arrow[d, hook, "G/K\otimes i"'] & X \arrow[d] \arrow[ddr, dashed] & \\
G/K \otimes B \arrow[r] \arrow[drr, dashed] & Y \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & H\backslash G \pitchfork Z
\end{tikzcd}
\qquad
\begin{tikzcd}
(G/K \otimes A)^H \arrow[r] \arrow[d, hook, "(G/K\otimes i)^H"'] & X^H \arrow[d] \arrow[ddr, dotted] & \\
(G/K \otimes B)^H \arrow[r] \arrow[drr, dotted] & Y^H & \\
& & Z
\end{tikzcd}$$

We want to show that the right-hand solid square is a pushout square in **Set**. Suppose we have maps $X^H \to Z$ and $(G/K \otimes B)^H \to Z$ (dotted) that make the outer square on the right commute. Then the adjoints $X \to H\backslash G \pitchfork Z$ and $G/K \otimes B \to H\backslash G \pitchfork Z$ (dashed) of these maps under the adjunction $(-)^H \dashv H\backslash G \pitchfork -$, dashed in the diagram on the left, make the outer square commute in that diagram. By the universal property of the pushout in $[\mathbf{B}G, \mathbf{Set}]$, there is then a unique map $Y \to H\backslash G \pitchfork Z$ commuting with these maps, and its adjoint $Y^H \to Z$ is the required map witnessing that the right-hand square is a pushout.

Finally, for (iii), we see that for $A \in \mathcal{C}$, the induced map factors as

$$(G/K)^H \otimes A \cong (G/K)^H \times A \cong (G/K)^H \times (\operatorname{triv} A)^H \cong (G/K \times \operatorname{triv} A)^H \cong (G/K \otimes A)^H,$$

where the third isomorphism follows from the fact that limits (fixed points) commute with limits (products). □

**Lemma 5.2.5** · *The adjunction $\mathrm{ev}_{G/e} \dashv \Phi$ is a Quillen adjunction between $[\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}, \mathcal{C}]$ with the projective model structure of Theorem 2.4.8 and $[\mathbf{B}G, \mathcal{C}]$ with the $\mathcal{F}$-model structure of Proposition 5.2.3, assuming these model structures exist for the model category $\mathcal{C}$.*

*Proof.* To show that $\mathrm{ev}_{G/e} \dashv \Phi$ is a Quillen adjunction, it suffices by Lemma 2.3.2 to show that $\Phi$ preserves fibrations and acyclic fibrations. A $G$-map $f : X \to Y$ is sent by $\Phi$ to the natural transformation $\alpha$ from the functor $G/H \mapsto X^H$ to $G/H \mapsto Y^H$ with $\alpha_{G/H} = f^H : X^H \to Y^H$ for all $H \in \mathcal{F}$. In the $\mathcal{F}$-model structure on $[\mathbf{B}G, \mathcal{C}]$, the map $f : X \to Y$ is a fibration or acyclic fibration precisely when $f^H : X^H \to Y^H$ is for all $H \in \mathcal{F}$, and in the projective model structure on $[\mathbf{Orb}^{\mathrm{op}}_{\mathcal{F}}, \mathcal{C}]$, the map $\alpha$ is a fibration or acyclic fibration if and only if $\alpha_{G/H} = f^H$ is for all $H \in \mathcal{F}$. The functor $\Phi$ thus indeed preserves these classes. □

**Theorem 5.2.6** (Elmendorf [23, Theorem 2.10]) · *Let $\mathcal{C}$ be a cofibrantly generated model category and $\mathcal{F}$ a collection of subgroups of $G$. If the fixed-point functors $(-)^H : [\mathbf{B}G, \mathcal{C}] \to \mathcal{C}$ satisfy the cellularity conditions for all $H \in \mathcal{F}$, then the adjunction $\mathrm{ev}_{G/e} \dashv \Phi$ is a Quillen equivalence between $[\mathbf{Orb}^{\mathrm{op}}_{\mathcal{F}}, \mathcal{C}]$ with the projective model structure of Theorem 2.4.8 and $[\mathbf{B}G, \mathcal{C}]$ with the $\mathcal{F}$-model structure of Proposition 5.2.3.*

To prove Elmendorf's theorem, we will use the following lemma.

**Lemma 5.2.7** ([3, Lemma 1.3.12]) · *Let $F \dashv U$ be a Quillen adjunction of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between model categories. If $U$ creates weak equivalences, then $F \dashv U$ is a Quillen equivalence if and only if for every cofibrant object $X$ of $\mathcal{C}$, the adjunction unit $\eta_X : X \to UFX$ is a weak equivalence.*

*Proof.* We want to show that the statement 'a map $f^\flat : X \to UY$ in $\mathcal{C}$ is a weak equivalence if and only if its adjoint $f^\sharp : FX \to Y$ is a weak equivalence in $\mathcal{D}$ for cofibrant $X$ in $\mathcal{C}$ and fibrant $Y$ in $\mathcal{D}$' is equivalent to the adjunction unit $\eta_X : X \to UFX$ being a weak equivalence for all cofibrant objects $X$ of $\mathcal{C}$. One direction follows from the commuting diagram

$$\begin{tikzcd} X \arrow[d, "\eta_X"'] \arrow[dr, "f^\flat"] & \\ UFX \arrow[r, "Uf^\sharp"'] & UY \end{tikzcd}$$

and the two-out-of-three property for weak equivalences, using the fact that $U$ creates weak equivalences. The converse direction follows by choosing the weak equivalence $r_{FX} : FX \to RFX$ given by fibrant replacement for $f^\sharp$ (and thus $Y = RFX$) in the diagram above. □

*Proof (of Elmendorf's theorem 5.2.6).* By Lemma 5.2.7, it suffices to show that the adjunction unit $\eta_X : X \to \Phi \circ \mathrm{ev}_{G/e}(X)$ is a weak equivalence for all cofibrant objects $X$. We will actually show that $\eta_X$ is an isomorphism for such $X$. Since every cofibrant object is a retract of an $I$-cell complex by Proposition 2.4.6, where $I$ is the set of generating cofibrations, and isomorphisms are closed under retracts, it suffices to check in the case that $X$ is an $I$-cell complex. Then there is an ordinal $\lambda > 0$ and a $\lambda$-sequence $Y : \lambda \to [\mathbf{Orb}^{\mathrm{op}}_{\mathcal{F}}, \mathcal{C}]$ whose transfinite composition is the map $\emptyset \to X$ (that is, with $Y_0 = \emptyset$ and $\operatorname{colim} Y = X$) where the maps $Y_\alpha \to Y_{\alpha+1}$ for $\alpha + 1 < \lambda$ are pushouts

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{Orb}^{\mathrm{op}}_{\mathcal{F}}}(G/K, -) \otimes A \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathbf{Orb}^{\mathrm{op}}_{\mathcal{F}}}(G/K,-)\otimes f"'] & Y_\alpha \arrow[d] \\ \mathrm{Hom}_{\mathbf{Orb}^{\mathrm{op}}_{\mathcal{F}}}(G/K, -) \otimes B \arrow[r] & Y_{\alpha+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{5.1}$$

for some $G/K \in \mathbf{Orb}_{\mathcal{F}}$ and a generating cofibration $f : A \to B$ of $\mathcal{C}$.

We will first show that the unit $\eta$ is an isomorphism at the free diagrams $\mathrm{Hom}_{\mathbf{Orb}^{\mathrm{op}}_{\mathcal{F}}}(G/K, -) \otimes A$ on all objects $A$. For any object $A$ of $\mathcal{C}$, there are isomorphisms

$$\mathrm{ev}_{G/e}(\mathrm{Hom}_{\mathbf{Orb}^{\mathrm{op}}_{\mathcal{F}}}(G/K, -) \otimes A) \cong (G/K)^e \otimes A \cong G/K \otimes A$$

of $G$-objects (with the canonical left $G$-action on the indexing sets of the latter objects) using Lemma 5.1.5. The unit $\eta_Z$ at a diagram $Z$, itself being a natural transformation, has components

$\eta_{Z,G/H} : Z(G/H) \to (\mathrm{ev}_{G/e} Z)^H$ induced by the universal property of the fixed-point object from the map $Z(\widetilde{e}) : Z(G/H) \to Z(G/e)$. The unit component

$$\mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}}(G/K, G/H) \otimes A \to \Phi \circ \mathrm{ev}_{G/e}(\mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}}(G/K, -) \otimes A)(G/H) \cong \Phi(G/K \otimes A)$$

at $\mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}}(G/K, -) \otimes A$ and $G/H$ is then equal to the map

$$\mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}}(G/K, G/H) \otimes A \cong (G/K)^H \otimes A \to (G/K \otimes A)^H$$

of (iii) of the cellularity conditions, which is an isomorphism.

The pushouts of (5.1) are preserved by the left adjoint $\mathrm{ev}_{G/e}$ and by $\Phi$ because of (ii) of the cellularity conditions. Since $\Phi$ is seen to preserve the initial object, the unit $\eta_{Y_0}$ at $Y_0 = \emptyset$ is an isomorphism. By transfinite induction, it now follows that $\eta_X$ is an isomorphism: in the successor case, if $\eta_{Y_\alpha}$ is an isomorphism, we see from the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}}(G/K, -) \otimes A \arrow[rr] \arrow[dr, "\eta", "\cong"'] \arrow[dd, "\mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}}(G/K,-)\otimes f"'] & & Y_\alpha \arrow[dr, "\eta", "\cong"'] \arrow[dd] & \\
& \Phi(G/K \otimes A) \arrow[rr] \arrow[dd] & & \Phi \circ \mathrm{ev}_{G/e}(Y_\alpha) \arrow[dd] \\
\mathrm{Hom}_{\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}}(G/K, -) \otimes B \arrow[rr] \arrow[dr, "\eta", "\cong"'] & & Y_{\alpha+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "\eta", "\cong"'] & \\
& \Phi(G/K \otimes B) \arrow[rr] & & \Phi \circ \mathrm{ev}_{G/e}(Y_{\alpha+1}) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

that $\eta_{Y_{\alpha+1}}$ is an isomorphism since $Y_{\alpha+1}$ and $\Phi \circ \mathrm{ev}_{G/e}(Y_{\alpha+1})$ are pushouts of isomorphic diagrams. The colimit case follows from (i) of the cellularity conditions. □

**Corollary 5.2.8** · *The homotopy categories of the contravariant orbit diagram category* $[\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}, \mathcal{C}]$ *and the category* $[\mathbf{B}G, \mathcal{C}]$ *of $G$-objects in $\mathcal{C}$ with respect to the projective and $\mathcal{F}$-model structures respectively are equivalent.*

*Proof.* Directly from Theorem 5.2.6 and Proposition 2.3.11. □

Elmendorf originally proved this for topological spaces in [6] (for actions of compact Lie groups) by directly showing that the homotopy categories are equivalent, without using Quillen's theory of model categories. According to Blumberg [3, p. 9], 'until the mid-1990s (...), many homotopy theorists avoid them [model categories], thinking of them as formal gobbledygook'. In [23], Stephan generalised Elmendorf's theorem to other cofibrantly generated model categories and presented the result in the language of model categories.

The following application, also presented in [6], illustrates the utility of approaching $G$-spaces from the perspective of orbit diagrams, enabled by Elmendorf's theorem.

*Example* 5.2.9 ([6, § 2], [3, Definition 1.3.14]) · Let $\mathcal{F}$ be a family of subgroups of a group $G$. If $X$ is a $G$-space, then the *isotropy group* (also called the *stabiliser subgroup*) of a point $x \in X$ is the subgroup $G_x := \{ g \in G \mid gx = x \}$. The $G$-space $X$ is called *$\mathcal{F}$-isotropic* if the family $\mathcal{F}$ contains all isotropy groups $G_x$ for points $x \in X$. A *classifying space* of $\mathcal{F}$ is a $G$-space $E\mathcal{F}$ such that $E\mathcal{F}$ is $\mathcal{F}$-isotropic and for every $\mathcal{F}$-isotropic $G$-space $X$, there is a unique equivariant map $X \to E\mathcal{F}$ up to homotopy. Using Elmendorf's theorem, we can obtain such a classifying space $E\mathcal{F}$ from a contravariant orbit diagram $F : \mathbf{Orb}_G \to \mathbf{Top}$ with

$$F : G/H \mapsto \begin{cases} * & \text{if } H \in \mathcal{F}, \\ \emptyset & \text{if } H \notin \mathcal{F}. \end{cases}$$

Applying $\mathrm{ev}_{G/e}$ to the cofibrant replacement of this diagram $F$ with respect to the projective model structure on $[\mathbf{Orb}_G^{\mathrm{op}}, \mathbf{Top}]$, we find a classifying space for the family $\mathcal{F}$. ◊

Given a weak equivalence $f : X \to Y$ between fibrant–cofibrant $G$-spaces in the $\mathcal{F}$-model structure on $[\mathbf{B}G, \mathbf{Top}]$, the induced maps $f_H : X_H \to Y_H$ on orbit spaces are weak equivalences for all $H \in \mathcal{F}$. Indeed, such a weak equivalence $f$ has an inverse $g : Y \to X$ up to $G$-homotopy by the Whitehead theorem for model categories, so there is a $G$-homotopy $h : X \times \mathrm{triv}[0,1] \to Y$ such that the left-hand diagram

$$\begin{tikzcd} X \arrow[r, "{(-,0)}"] \arrow[dr, "gf"'] & X \times \mathrm{triv}[0,1] \arrow[d, "h"] & X \arrow[l, "{(-,1)}"'] \arrow[dl, "1_X"] \\ & Y & \end{tikzcd} \quad \overset{(-)_H}{\longmapsto} \quad \begin{tikzcd} X_H \arrow[r, "{(-,0)}"] \arrow[dr, "g_H f_H"'] & X_H \times [0,1] \arrow[d, "h_H"] & X_H \arrow[l, "{(-,1)}"'] \arrow[dl, "1_{X_H}"] \\ & Y_H & \end{tikzcd}$$

commutes. Since $(X \times \mathrm{triv}[0,1])_H \cong X_H \times [0,1]$, applying the orbit functor $(-)_H$ to the left-hand diagram results in the right-hand diagram, showing that it induces a homotopy $h_H : X_H \times [0,1] \to Y_H$ from $g_H f_H$ to $1_{X_H}$. Similarly, we show that the converse composite $f_H g_H$ is homotopic to $1_{Y_H}$, and see that $g_H$ is a homotopy inverse of $f_H$. Again by the Whitehead theorem for model categories, we then conclude that the induced map $f_H$ is a weak equivalence.

The converse statement does not hold, however: a $G$-map between fibrant–cofibrant $G$-spaces in the $\mathcal{F}$-model structure that induces weak equivalences on $H$-orbits for all $H \in \mathcal{F}$ need not be a weak equivalence in the $\mathcal{F}$-orbit model structure. A counterexample, due to Tom Goodwillie, is presented in [7, p. 1132]. This article explores when maps that induce weak equivalences on orbits also induce weak equivalences on fixed-points, and is the subject of the next section.

## 5.3 Via orbits

Dual to the functor $\Phi : [\mathbf{B}G, \mathcal{C}] \to [\mathbf{Orb}_{\mathcal{F}}^{\mathrm{op}}, \mathcal{C}]$, we can define a functor $\Psi : [\mathbf{B}G^{\mathrm{op}}, \mathcal{C}] \to [\mathbf{Orb}_{\mathcal{F}}, \mathcal{C}]$, again by currying the functor $[\mathbf{B}G^{\mathrm{op}}, \mathcal{C}] \times \mathbf{Orb}_{\mathcal{F}} \to \mathcal{C}$ which sends a pair $(X, G/H)$ to the orbit object $X_H$ and a pair of maps $f : X \to Y$ and $\widetilde{g} : G/H \to G/K$ to the unique map $X_H \to Y_K$ for which the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "h^*"] \arrow[r, shift right, "(h')^*"'] & X \arrow[r, "q_H^X"] \arrow[d, "f"'] & X_H \arrow[d, dashed] \\ & Y \arrow[r, "g^*"'] & Y \arrow[r, "q_K^Y"'] & Y_K \end{tikzcd}$$

commutes for all $h, h' \in H$, where $q_H^X$ and $q_K^Y$ are the orbit quotient maps. Note that we need $X$ and $Y$ to be *right* $G$-objects to use the automorphism of $Y$ induced by $g$ in this diagram. A similar construction can be done for left $G$-objects (which are just right $G^{\mathrm{op}}$-objects), but then the map induced by $g^{-1}$ should be used.

In the opposite direction, precomposition with the functor $i^{\mathrm{op}} : \mathbf{B}G^{\mathrm{op}} \hookrightarrow \mathbf{Orb}_{\mathcal{F}}$ seeing $G/e$ as a right $G$-object in $\mathbf{Orb}_{\mathcal{F}}$, with $g \mapsto \widetilde{g}$, defines a functor $\mathrm{ev}_{G/e} := (i^{\mathrm{op}})^* : [\mathbf{Orb}_{\mathcal{F}}, \mathcal{C}] \to [\mathbf{B}G^{\mathrm{op}}, \mathcal{C}]$.

**Lemma 5.3.1** · *The functor* $\mathrm{ev}_{G/e} : [\mathbf{Orb}_{\mathcal{F}}, \mathcal{C}] \to [\mathbf{B}G^{\mathrm{op}}, \mathcal{C}]$ *restricting to* $G/e$ *is right adjoint to the functor* $\Psi : [\mathbf{B}G^{\mathrm{op}}, \mathcal{C}] \to [\mathbf{Orb}_{\mathcal{F}}, \mathcal{C}]$.

*Proof.* Entirely dual to Lemma 5.2.1, we recognise $\Psi X$ as a left Kan extension of the $G$-object $X$ along $i^{\mathrm{op}} : \mathbf{B}G^{\mathrm{op}} \hookrightarrow \mathbf{Orb}_{\mathcal{F}}$, showing that $\Psi$ is left adjoint to $\mathrm{ev}_{G/e}$. □

Dual to the $\mathcal{F}$-model structure on left $G$-simplicial sets of Proposition 5.2.3 where weak equivalences and fibrations are created by the fixed-point functors, Erdal and Güçlükan İlhan show that there is a model structure on the category of right $G$-simplicial sets where weak equivalences and cofibrations are created by the orbit functors.

**Theorem 5.3.2** ([7, Theorem 2]) · *Let* $\mathcal{F}$ *be a family of subgroups of* $G$. *There is a model structure on the category* $[\mathbf{B}G^{\mathrm{op}}, \mathbf{sSet}]$ *of right* $G$*-simplicial sets where weak equivalences and cofibrations are created by all orbit functors* $(-)_H : [\mathbf{B}G^{\mathrm{op}}, \mathbf{sSet}] \to \mathbf{sSet}$ *for* $H \in \mathcal{F}$.

We will call this model structure the *$\mathcal{F}$-orbit model structure.* To prove the existence of this model structure on $G$-simplicial sets, Erdal and Güçlükan İlhan use the results from [11] to *left-induce* the model structure on $[\mathbf{B}G^{\mathrm{op}}, \mathbf{sSet}]$ from the injective model structure on the orbit diagram category $[\mathbf{Orb}_{\mathcal{F}}, \mathbf{sSet}]$. In a left-induced model structure, the weak equivalences and cofibrations are created by a left adjoint functor (in this case $\Psi$). Left inducing model structures is technically more involved than right inducing; to apply the theory of [11], it is necessary to restrict to *accessible* model categories (such as **sSet**), which are in particular locally presentable. Since the category of topological spaces is not locally presentable, the left-induction argument does not provide an $\mathcal{F}$-orbit model structure for topological spaces. By the Quillen equivalence between **sSet** and **Top** of Theorem 3.5.3, we can at least be satisfied by the $\mathcal{F}$-orbit model structure for simplicial sets.

Generalisations of the $\mathcal{F}$-orbit model structure to $G$-objects in other categories than simplicial sets are possible, however. Erdal and Güçlükan İlhan show that the category of *$\Delta$-generated topological spaces*, a 'very convenient' subcategory of **Top**, also admits the $\mathcal{F}$-orbit model structure, and discuss conditions for general model categories to admit this model structure [7, § 3.2].

The following lemma characterises the cofibrations in the $\mathcal{F}$-orbit model structure.

**Lemma 5.3.3** ([7, p. 1136]) · *A simplicial $G$-map $f : X \to Y$ is a cofibration in the $\mathcal{F}$-orbit model structure of Theorem 5.3.2 if and only if $f$ is a levelwise injective simplicial map.*

*Proof.* If $f$ is a cofibration in the $\mathcal{F}$-model structure, then the induced map $f_e = f : A_e \cong A \to B_e \cong B$ on the orbits of the trivial subgroup, which are isomorphic to the original objects, is a cofibration of simplicial sets, and hence levelwise injective by Theorem 3.5.2. Conversely, suppose $f$ is levelwise injective. Write $[x]_H$ for the image of an $n$-simplex $x$ of $X$ under the quotient map $X \to X_H$ for $H \in \mathcal{F}$ (and similarly for simplices of $Y$); then $f_H : X_H \to Y_H$ is the map with $(f_H)_n([x]_H) = [f_n(x)]_H$ for all $x$. If $[f_n(x)]_H = [f_n(x')]_H$, then there exists $h \in H$ such that $f_n(x) = f_n(x') \cdot h = f_n(x' \cdot h)$. By injectivity of $f$, we see that $x = x' \cdot h$, whence $[x]_H = [x']_H$ and $f_H$ is levelwise injective, and thus a cofibration of simplicial sets. □

We now discuss the dual to Elmendorf's theorem. The setup is formally dual: the adjunction $\Psi \dashv \mathrm{ev}_{G/e}$, dual to $\mathrm{ev}_{G/e} \dashv \Phi$, is also a Quillen adjunction.

**Lemma 5.3.4** · *The adjunction $\Psi \dashv \mathrm{ev}_{G/e}$ is a Quillen adjunction between $[\mathbf{B}G^{\mathrm{op}}, \mathbf{sSet}]$ with the $\mathcal{F}$-orbit model structure of Theorem 5.3.2 and $[\mathbf{Orb}_{\mathcal{F}}, \mathbf{sSet}]$ with the injective model structure of Theorem 2.4.10.*

*Proof.* Dual to the proof of Lemma 5.2.5, using the fact that weak equivalences and cofibrations are generated by $\Psi$ in the model structure on $[\mathbf{B}G^{\mathrm{op}}, \mathbf{sSet}]$ and are pointwise weak equivalences and cofibrations in the injective model structure on $[\mathbf{Orb}_{\mathcal{F}}, \mathbf{sSet}]$. □

The duality between the model structures via fixed points and via orbits fails when we consider Elmendorf's theorem 5.2.6, however: the Quillen adjunction $\Psi \dashv \mathrm{ev}_{G/e}$ is in general not a Quillen equivalence. To give a counterexample, Erdal and Güçlükan İlhan use the following model categorical lemma, which is dual to Lemma 5.2.7. The proof given in [7] is different from the dual of the proof of Lemma 5.2.7 we presented.[1]

**Lemma 5.3.5** ([7, Lemma 1]) · *Let $F \dashv U$ be a Quillen adjunction of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between model categories. If $F$ creates weak equivalences, then $F \dashv U$ is a Quillen equivalence if and only if for every fibrant object $Y$ of $\mathcal{D}$, the adjunction counit $\varepsilon_Y : FUY \to Y$ is a weak equivalence.*

We now give a counterexample showing that the Quillen adjunction $\Psi \dashv \mathrm{ev}_{G/e}$ is not a Quillen equivalence for every non-trivial finite group $G$, generalising the counterexample for $G = C_2$ given in [7, p. 1137].

[1] In the statement of Lemma 5.3.5, we correct the minor mistake that $Y$ should be an object of $\mathcal{D}$ instead of $\mathcal{C}$.

*Example* 5.3.6 · Let $G$ be a non-trivial finite group. We want to use Lemma 5.3.5 to show that $\Psi \dashv \mathrm{ev}_{G/e}$ is not a Quillen equivalence. Thus, we need to construct a fibrant object of $[\mathbf{Orb}_G, \mathbf{sSet}]$ with the injective model structure such that the counit at that object is not a weak equivalence.

To that end, pick a set $X$ such that there is no transitive group action of $G$ on $X$, for example with one element more than $\#G$. Define an orbit diagram $F : \mathbf{Orb}_G \to \mathbf{sSet}$ by taking $F(G/e)$ to be the discrete simplicial set on $X$ (that is, using the construction of Example 3.2.3) and $F(G/H) := *$, the terminal simplicial set, for non-trivial subgroups $H$ of $G$. All maps in $\mathbf{Orb}_G$ into $G/e$ are endomorphisms, and we define their image under $F$ to be the identity on $X$; the image of all other maps must be the unique map into the terminal simplicial set.

Let $\tilde{F} : \mathbf{Orb}_G \to \mathbf{sSet}$ be the fibrant replacement of $F$ in $[\mathbf{Orb}_G, \mathbf{sSet}]$ with the injective model structure. Since there is a natural weak equivalence $F \xRightarrow{\sim} \tilde{F}$ from $F$ to its fibrant replacement, we have weak equivalences $F(G/H) \xrightarrow{\sim} \tilde{F}(G/H)$ of simplicial sets for all $G/H \in \mathbf{Orb}_G$. In particular, $\tilde{F}(G/e)$ is weakly equivalent to the discrete simplicial set on $X$ and $\tilde{F}(G/G)$ to the terminal simplicial set, and we have isomorphisms $\pi_0|\tilde{F}(G/e)| \cong \pi_0|F(G/e)| \cong X$ and $\pi_0|\tilde{F}(G/G)| \cong *$ of path components. We want to show that the adjunction counit $\varepsilon_{\tilde{F}} : \Psi \circ \mathrm{ev}_{G/e}(\tilde{F}) \Rightarrow \tilde{F}$ at $\tilde{F}$ is not a weak equivalence in $[\mathbf{Orb}_G, \mathbf{sSet}]$.

If $\varepsilon_{\tilde{F}}$ is a weak equivalence, then its component $(\mathrm{ev}_{G/e}\tilde{F})_G \xrightarrow{\sim} \tilde{F}(G/G)$ at $G/G$ is a weak equivalence in $\mathbf{sSet}$, which means that there is a weak homotopy equivalence $|(\mathrm{ev}_{G/e}\tilde{F})_G| \xrightarrow{\sim} |\tilde{F}(G/G)|$ in $\mathbf{Top}$. We then have an isomorphism $\pi_0|(\mathrm{ev}_{G/e}\tilde{F})_G| \cong \pi_0|\tilde{F}(G/G)| \cong *$. The composite functor $\pi_0 \circ |-| : \mathbf{sSet} \to \mathbf{Set}$ has a right adjoint which sends a set $Y$ to the singular set of $Y$ with the discrete topology, with $n$-simplices $\mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y) \cong Y$, so it commutes with colimits. We thus find $(\pi_0|\mathrm{ev}_{G/e}\tilde{F}|)_G \cong \pi_0|(\mathrm{ev}_{G/e}\tilde{F})_G| \cong *$. (Here we use the notation $\pi_0|Z|$ for the $G$-space $\pi_0 \circ |-| \circ Z$ if $Z : \mathbf{B}G^{\mathrm{op}} \to \mathbf{sSet}$ is a $G$-simplicial set.) Since $\pi_0|\tilde{F}(G/e)| \cong X$, however, that would mean there is a transitive $G$-action on $X$, which is a contradiction. ◊

Dualising this counterexample to try to invalidate Elmendorf's theorem – by defining a contravariant orbit diagram $F$ with $F(G/e) = X$ and $F(G/H) = \emptyset$ for non-trivial $H$, taking the cofibrant replacement in the projective model structure and looking at the unit in $G/G$ – does not work. There are spaces weak homotopy equivalent to $X$ which admit a free $G$-action, and thus have no $G$-fixed points. For example, the $X$-fold coproduct $\coprod_X G$ of $\#G + 1$ copies of $G$ with the indiscrete topology is homotopy equivalent to $X$ as a discrete space, and admits a free $G$-action where $G$ acts by multiplication on each of the coproduct factors.

Erdal and Güçlükan İlhan use the model structure 'via orbits' of Theorem 5.3.2 to give a model categorical criterion for when equivariant maps that induce weak equivalences on orbits also induce weak equivalences on fixed points. This is the main application of the $\mathcal{F}$-orbit model structure they present.

Between the categories of left and right $G$-simplicial sets, there is an adjoint equivalence

$$\begin{tikzcd} {[\mathbf{B}G, \mathbf{sSet}]} \arrow[r, shift left, "\mathrm{id}_-"] \arrow[r, phantom, "\perp"] & {[\mathbf{B}G^{\mathrm{op}}, \mathbf{sSet}]} \arrow[l, shift left, "\mathrm{id}^-"] \end{tikzcd}$$

under which a (left or right) $G$-object is sent to itself with the reversed (right or left) $G$-action.

**Proposition** 5.3.7 ([7, Proposition 3]) · *Let $\mathcal{F}$ be a family of subgroups of $G$. The adjunction* $\mathrm{id}_- \dashv \mathrm{id}^-$ *is a Quillen adjunction between* $[\mathbf{B}G, \mathbf{sSet}]$ *with the $\mathcal{F}$-model structure of Proposition 5.2.3 and* $[\mathbf{B}G^{\mathrm{op}}, \mathbf{sSet}]$ *with the $\mathcal{F}$-orbit model structure of Theorem 5.3.2.*

*Proof.* We show that $\mathrm{id}_-$ preserves cofibrations and acyclic cofibrations. The (acyclic) cofibrations in the $\mathcal{F}$-model structure are retracts of transfinite compositions of pushouts of generating (acyclic) cofibrations by Proposition 2.4.6. The generating cofibrations are the maps $G/H \otimes \partial\Delta^n \to G/H \otimes \Delta^n$ induced by the boundary inclusions, and the generating acyclic cofibrations are the maps $G/H \otimes \Lambda^n_k \to G/H \otimes \Delta^n$ induced by the horn inclusions for all $H \in \mathcal{F}$. Since the boundary and horn inclusions, when considered as $G$-equivariant maps with respect to the trivial actions, are cofibrations in the $\mathcal{F}$-orbit model structure, it follows that the generating cofibrations and generating acyclic cofibrations of the $\mathcal{F}$-model structure are respectively cofibrations and acyclic cofibrations in the $\mathcal{F}$-orbit model structure by Lemma 5.3.3, since they are levelwise injective simplicial maps.

We have thus seen that $\mathrm{id}_{-}$ takes generating (acyclic) cofibrations to (acyclic) cofibrations. The cofibrations are closed under retracts by (MC2), stable under pushouts by Lemma 2.1.15, and also closed under transfinite compositions (which follows from the fact that they are exactly the maps with the left-lifting property with respect to acyclic fibrations, see [20, Lemma 11.1.4]). Hence, the left adjoint $\mathrm{id}_{-}$, which preserves colimits and thus pushouts and transfinite composition, takes all (acyclic) cofibrations to (acyclic) cofibrations, and is thus a left Quillen functor. □

We now spell out the Whitehead theorem for the $\mathcal{F}$-orbit model structure. A simplicial $G$-homotopy between simplicial $G$-maps $f, g : X \to Y$ is a simplicial $G$-map $h : X \times \operatorname{triv} \Delta^1 \to Y$ such that the diagram

$$\begin{tikzcd} X \cong X \times \operatorname{triv}\Delta^0 \arrow[r, "1_X \times \Delta^- d^1"] \arrow[dr, "f"'] & X \times \operatorname{triv}\Delta^1 \arrow[d, "h"] & X \times \operatorname{triv}\Delta^0 \cong X \arrow[l, "1_X \times \Delta^- d^0"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

commutes. Correspondingly, a simplicial $G$-homotopy equivalence is a simplicial $G$-map with an inverse up to simplicial $G$-homotopy. Since every object in the $\mathcal{F}$-orbit model structure for simplicial sets is cofibrant (because all simplicial sets are cofibrant), the Whitehead theorem takes the following form.

**Proposition 5.3.8** (Whitehead [7, Corollary 2]) · *Let $X$ and $Y$ be fibrant in the $\mathcal{F}$-orbit model structure on $[\mathbf{B}G^{\mathrm{op}}, \mathbf{sSet}]$. A simplicial $G$-map $f : X \to Y$ induces weak equivalences $f_H : X_H \to Y_H$ on $H$-orbits for all $H \in \mathcal{F}$ if and only if $f$ is a simplicial $G$-homotopy equivalence.*

Using the Whitehead theorem for the $\mathcal{F}$-model structure, we obtain the following criterion for when $G$-equivariant maps that induce weak equivalences on $H$-orbits for all $H \in \mathcal{F}$ also induce weak equivalences, and even homotopy equivalences, on $H$-fixed points for all $H \in \mathcal{F}$, namely when the domain and codomain are fibrant in the $\mathcal{F}$-orbit model structure.

**Corollary 5.3.9** ([7, Corollary 1]) · *If $X$ and $Y$ are fibrant in the $\mathcal{F}$-orbit model structure on $[\mathbf{B}G^{\mathrm{op}}, \mathbf{sSet}]$ and $f : X \to Y$ induces weak equivalences $f_H : X_H \to Y_H$ for all $H \in \mathcal{F}$, then $f^H : X^H \to Y^H$ is a homotopy equivalence, and in particular a weak equivalence, for all $H \in \mathcal{F}$.*

The argument is almost entirely dual to that of the dual statement discussed in § 5.2, only replacing topological spaces by simplicial sets.

*Remark* 5.3.10 · The statement of Corollary 5.3.9 is slightly stronger than [7, Corollary 1], whose proof uses a different strategy. In that article, it is shown that such a map $f$ induces weak equivalences (instead of homotopy equivalences) on $H$-fixed points for all $H \in \mathcal{F}$ using the Quillen adjunction of Proposition 5.3.7 and Ken Brown's lemma 2.3.5. ◊

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