Equivariant homotopy theory via orbits

Final presentation

Splinter Suidman

supervisor: Dr Magdalena Kędziorek

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Project overview

- Erdal & Güçlükan İlhan (2019), ‘A model structure via orbit spaces for equivariant homotopy’
- Algebraic topology
- Category theory


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A model structure via orbit spaces for equivariant homotopy

Mehmet Akif Erdal[1,2] · Aslı Güçlükan İlhan[3]

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Abstract
Let G be discrete group and $\mathcal{F}$ be a collection of subgroups of G. We show that there exists a left induced model structure on the category of right G-simplicial sets, in which the weak equivalences and cofibrations are the maps that induce weak equivalences and cofibrations on H-orbits for all H in $\mathcal{F}$. This gives a model categorical criterion for maps that induce weak equivalences on H-orbits to be weak equivalences in the $\mathcal{F}$-model structure.

Keywords Equivariant homotopy · Orbit space · Model structure

Mathematics Subject Classification 55U40 · 55U35

1 Introduction

Let $\mathcal{F}$ be a collection of subgroups of a given discrete group G which is closed under conjugation and taking subgroups. By a space, we mean a simplicial set and by a map we mean a simplicial map. There is a well-known model structure on the category ${}_G\mathbf{S}$ of G-spaces, called the $\mathcal{F}$-model structure, in which the weak equivalences and fibrations are maps that induce weak equivalences and fibrations on H-fixed points for

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Mehmet Akif Erdal
merdal@fen.bilkent.edu.tr

Aslı Güçlükan İlhan
asli.ilhan@deu.edu.tr

1 Department of Mathematics, Bilkent University, 06800 Ankara, Turkey
2 Present Address: Department of Mathematics, Yeditepe University, 34755 Istanbul, Turkey
3 Department of Mathematics, Dokuz Eylül University, 35400 Izmir, Turkey

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Outline

Category theory

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Abstract homotopy theory

Equivariant homotopy theory

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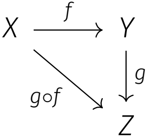
Category theory

- ‘Abstract nonsense’
- Mac Lane & Eilenberg, 1945
- Algebraic topology

Categories

- **Objects** $X, Y, Z, \dots$
- **Maps** (or morphisms) $f, g, h, \dots$ between objects
- Identity maps: $\mathrm{id}_X : X \to X$
- Composition:



- Composition associative and id_X identities

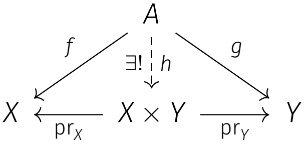
Categories: examples

category	*objects*	*maps*
Set	sets	functions
Grp	groups	group homomorphisms
Top	topological spaces	continuous maps
$\mathbf{Top}_*$	topological spaces with basepoint	basepoint-preserving continuous maps

If G is a group:

$\mathbf{B}G$	single object	group elements $g \in G$

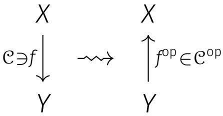
Categorically: products



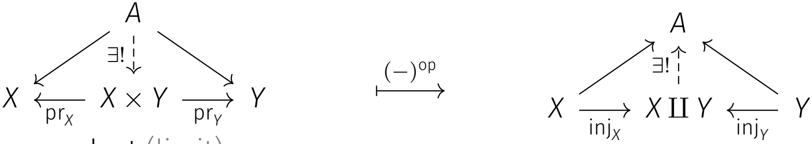
- In **Set**: Cartesian product
- In **Grp**: direct product
- In **Top**: product topology
- ‘Abstract nonsense’
  - $X \times Y \cong Y \times X$
- Example of a **limit**

Duality

- A category $\mathcal{C}$ has an **opposite category** $\mathcal{C}^{\mathrm{op}}$:
  - objects: same as $\mathcal{C}$
  - map $f^{\mathrm{op}} : Y \to X$ if $f : X \to Y$ in $\mathcal{C}$



- Theorems in category theory have **dual** versions



product (limit)

$X \times Y \cong Y \times X$

coproduct (colimit)

$X \amalg Y \cong Y \amalg X$

Functors: maps of categories

- **Functor** $F : \mathcal{C} \to \mathcal{D}$:
  - object X of $\mathcal{C} \overset{F}{\mapsto}$ object $F(X)$ of $\mathcal{D}$
  - map $f : X \to Y$ in $\mathcal{C} \overset{F}{\mapsto}$ map $F(f) : F(X) \to F(Y)$ of $\mathcal{D}$
  - preserving identity maps and composition
- Examples:
  - Forgetful functors $\mathbf{Grp} \to \mathbf{Set}$, $\mathbf{Top} \to \mathbf{Set}$, …
  - Free functors $\mathbf{Set} \to \mathbf{Grp}$, $\mathbf{Set} \to \mathbf{Top}$, …
  - Fundamental group $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$, $(X, x) \mapsto \pi_1(X, x)$,
    $f : X \to Y \;\mapsto\; f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$
- Category **Cat** of categories with functors as maps

Outline

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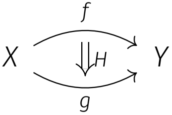
Abstract homotopy theory

Equivariant homotopy theory

Homotopy theory

- Category **Ho Top** of topological spaces and maps up to homotopy
- Recall: **homotopy** between $f, g : X \to Y$ is a map $H : X \times [0,1] \to Y$ with $H(-,0) = f$ and $H(-,1) = g$



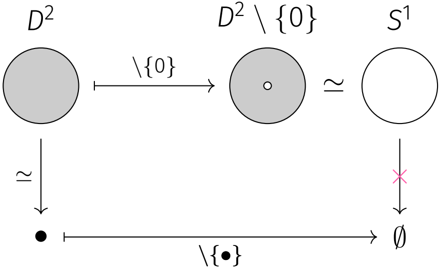
- Isomorphisms:
  - **Top**: homeomorphisms
  - **Ho Top**: homotopy equivalences (inverse up to homotopy)
- Importance of maps in categories

Topology vs homotopy theory

- Topology: If $f : X \to Y$ is a homeomorphism and $x \in X$, then

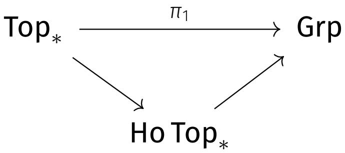
$$X \setminus \{x\} \cong Y \setminus \{f(x)\} \quad \in \mathbf{Top}$$

- Homotopy theory: in **Ho Top**,



- Removing a point is a topological but *not* a homotopical operation

- Fundamental group $\pi_1 : \textsf{Top}_* \to \textsf{Grp}$ is a **homotopy invariant**: homotopy equivalent spaces have isomorphic fundamental groups
- Alternatively: functor π_1 factors through $\textsf{Ho}\,\textsf{Top}_*$:



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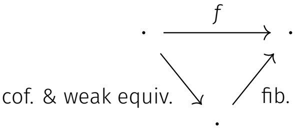
Abstract homotopy theory

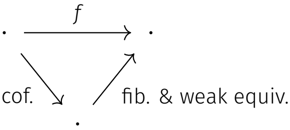
- Homotopy in different categories than **Top**
- Here: equivariant homotopy, of spaces with symmetries

Model structures

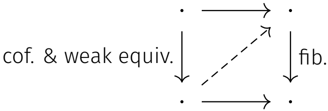
- Abstract homotopy theory
  - Category $\mathcal{C}$ with class of maps: *weak equivalences* resembling isomorphisms
  - Idea: formally invert weak equivalences of $\mathcal{C}$
- Solution (Quillen, 1967): *model structure* on $\mathcal{C}$, with classes of maps of $\mathcal{C}$:
  - *weak equivalences*
  - *fibrations*
  - *cofibrations*
  - ... satisfying certain axioms
- Homotopy category $\mathsf{Ho}\,\mathcal{C}$

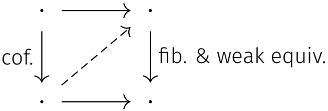
- Factorisation:





- Lifting:



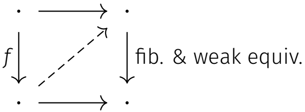


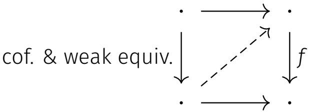
Duality in model categories

- Axioms are **self-dual**: model structure on $\mathcal{C}$ $\rightsquigarrow$ model structure on $\mathcal{C}^{op}$:

$\mathcal{C}^{op}$		$\mathcal{C}$
f^{op} weak equivalence	$\Longleftrightarrow$	f weak equivalence
f^{op} cofibration	$\Longleftrightarrow$	f fibration
f^{op} fibration	$\Longleftrightarrow$	f cofibration

- Theorems about model categories have dual versions
- Example: f cofibration $\Longleftrightarrow$ f has **left-lifting property** w.r.t. fibrations & weak equivalences





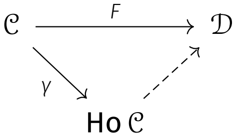
- Dual: f fibration $\Longleftrightarrow$ f has **right-lifting property** w.r.t. cofibrations & weak equivalences

Examples of model categories

category	*weak equivalences*	*cofibrations*	*fibrations*
any	isomorphisms	all maps	all maps
Set	all maps	surjective	injective
Top (Strøm)	homotopy equiv.	…	Hurewicz fib.
(Strøm (1972), 'The homotopy category is a homotopy category')			
Top (Quillen)	weak homotopy equiv.	…	Serre fib.

Homotopy category

- **Homotopy category Ho $\mathcal{C}$** of model category $\mathcal{C}$
- Functor $\gamma : \mathcal{C} \to \mathbf{Ho}\,\mathcal{C}$ inverting weak equivalences
- If $F : \mathcal{C} \to \mathcal{D}$ takes weak equivalences to isomorphisms (F is **homotopical**):



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Equivariant homotopy theory

- Homotopy theory of spaces (objects) with symmetries
- Group actions
- Model category of spaces with group actions

Group actions

- G-**action** of object X in category $\mathcal{C}$:
  - A map $g_* : X \to X$ for all $g \in G$
  - ... with $(g \cdot h)_* = g_* \circ h_*$ for all $g, h \in G$
  - ... and $e_* = \mathrm{id}_X$
- G-**equivariant map**: map $f : X \to Y$ in $\mathcal{C}$ that commutes with G-actions of X and Y:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ X & \xrightarrow[f]{} & Y \end{array}$$

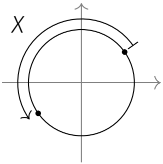
- Category $G\mathcal{C}$ of G-**objects** in $\mathcal{C}$ and equivariant maps
- Categorically:

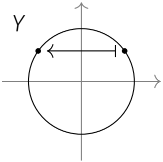
G-objects in $\mathcal{C}$	$\rightsquigarrow$	functors $\mathbf{B}G \to \mathcal{C}$
equivariant maps	$\rightsquigarrow$	natural transformations

Fixed points and orbits

- For a G-set X:
  - **Fixed-point set** $X^G = \{ x \in X \mid g_*(x) = x \ \ \forall g \in G \}$
  - **Orbit set** $X_G = X/{\sim_G}$ with $g_*(x) \sim_G x$
- For a G-space X:
  - **Fixed-point space**: subspace $X^G \hookrightarrow X$
  - **Orbit space**: quotient space $X_G/{\sim_G}$
- Generally: in a category $\mathfrak{C}$:
  - **Fixed-point object**: limit (when it exists)
  - **Orbit object**: colimit (when it exists)
  - Functors $(-)^G : G\mathfrak{C} \to \mathfrak{C}$ and $(-)_G : G\mathfrak{C} \to \mathfrak{C}$
  - Equivariant $f : X \to Y$ induces $f^G : X^G \to Y^G$ and $f_G : X_G \to Y_G$

- X: $C_2 = \mathbb{Z}/2\mathbb{Z}$ acting on S^1 by 180° rotation
    - Fixed-point space $X^{C_2} = \emptyset$
    - Orbit space $X_{C_2} \cong S^1$
- Y: C_2 acting on S^1 by reflection
    - Fixed-point space $Y^{C_2} = \{\text{north pole}\} \sqcup \{\text{south pole}\}$
    - Orbit space $Y_{C_2} \cong [0, 1]$
- Equivariant maps preserve fixed points, so no *equivariant* map $Y \to X$

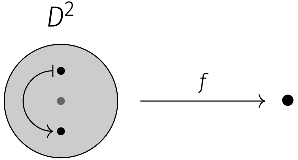




Equivariant homotopy theory: traditionally

- Homotopy of G-spaces (G finite): model structure on G**Top**
- A map $f : X \to Y$ in G**Top** is:
    - weak equivalence if $f^H : X^H \to Y^H$ weak equivalence in **Top** for all subgroups H of G
    - fibration if $f^H : X^H \to Y^H$ fibration in **Top** for all subgroups H
    - cofibration if it has the required lifting property
- Weak equivalences and fibrations are **created** by fixed-point functors $(-)^H : G\textbf{Top} \to \textbf{Top}$

Equivariant homotopy theory: example

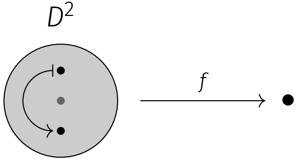


- C_2 acting on D^2 by 180° rotation around centre
- Unique equivariant map $f : D^2 \to \bullet$
  - $f^e = f : (D^2)^e = D^2 \to \bullet^e = \bullet$ weak equivalence
  - $f^{C_2} = \mathrm{id}_\bullet : (D^2)^{C_2} = \bullet \to \bullet^{C_2} = \bullet$ weak equivalence
  - So: f is a weak equivalence in C_2**Top**
  - In **Ho** C_2**Top**: $D^2 \cong \bullet$

Equivariant homotopy theory: via orbits

- Want: dual model structure on G**Top**, via orbits
- A map $f : X \to Y$ in $G\mathbf{Top}_{\text{orbits}}$ is:
  - weak equivalence if $f_H : X_H \to Y_H$ weak equivalence in **Top** for all subgroups H of G
  - cofibration if $f_H : X_H \to Y_H$ cofibration in **Top** for all subgroups H
  - fibration if it has the required lifting property

Orbits: example



- C_2 acting on D^2 by 180° rotation around centre
- Unique equivariant map $f : D^2 \to \bullet$
  - $f_e = f : (D^2)_e = D^2 \to \bullet_e = \bullet$ weak equivalence
  - $f_{C_2} = f : (D^2)_{C_2} \cong D^2 \to \bullet_{C_2} = \bullet$ weak equivalence
  - So: f would be a weak equivalence in $C_2\mathbf{Top}_{\text{orbits}}$
  - In $\mathbf{Ho}\, C_2\mathbf{Top}_{\text{orbits}}$: $D^2 \cong \bullet$

Model structure via orbits

- Model structure $G\mathbf{Top}_{\text{fixed points}}$ is **right-induced**:

$$\text{orbit diagrams} \underset{R}{\overset{\perp}{\rightleftarrows}} G\mathbf{Top}_{\text{fixed points}}$$

  - Equivalence of homotopy categories (Elmendorf, 1983)
- Idea: **left-induce** model structure $G\mathbf{Top}_{\text{orbits}}$ (Erdal & Güçlükan İlhan, 2019):

$$G\mathbf{Top}_{\text{orbits}} \overset{L}{\underset{\perp}{\rightleftarrows}} \text{orbit diagrams}$$

  - Left-inducing technically more involved (Hess, Kędziorek, Riehl & Shipley, 2017)
  - 'Nice' category of spaces: simplicial sets ($\mathrm{Ho}\,\mathrm{sSet} \simeq \mathrm{Ho}\,\mathrm{Top}$)
  - No equivalence of homotopy categories

Conclusion

- Duality in (model) categories
- Model structure via orbits possible, but harder
- Failure of duality