Homotopy category + Quillen functors

Today: Previously: axioms, examples, homotopy relation

- Homotopy category $\mathrm{Ho}(\mathcal{M})$: localisation of model category $\mathcal{M}$ w.r.t. weak equivalences
- Derived functors between homotopy categories

Throughout: $\mathcal{M}, \mathcal{N}$ model categories
$\mathcal{C}, \mathcal{D}$ categories

§ (Co)fibrant replacement

<u>Assume</u>: model categories have <u>functorial</u> factorisations.

Okay: most mod. cat's we care about have this, by small obj. arg.

<u>Def</u>. A <u>functorial factorisation</u> on $\mathcal{C}$ is a section of $d_1 = \circ : \mathrm{Fun}([2], \mathcal{C}) \to \mathrm{Fun}([1], \mathcal{C})$.

<u>Cor</u>. A model category $\mathcal{M}$ has an endofunctor Q with a natural w.e. $q : Q \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{M}}$ s.t. QX is cofibrant for every $X \in \mathcal{M}$. We call Q <u>cofibrant replacement</u>.

Dually: <u>fibrant replacement</u> $r : \mathrm{id}_{\mathcal{M}} \overset{\sim}{\Rightarrow} R$.

Proof. Factor
$$\begin{tikzcd} \emptyset \arrow[rr] \arrow[dr, hook] & & X \\ & QX \arrow[ur, two heads, "\sim"'] & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "\sim"'] & & * \\ & RX \arrow[ur, two heads] & \end{tikzcd}$$
$\square$

Lem. Q and R create weak equivalences.

Proof. $f: X \to Y$ in $\mathcal{M}$ $\rightsquigarrow$

$$\begin{tikzcd} QX \arrow[r, "q_X", "\sim"'] \arrow[d, "Qf"'] & X \arrow[d, "f"] \\ QY \arrow[r, "\sim", "q_Y"'] & Y \end{tikzcd}$$

$\square$

§ Homotopy category

Recall: homotopy relation $\sim$ on $\mathrm{Hom}_{\mathcal{M}}(A,X)$ equiv. rel. if A cof. and X fib, respected by composition.

Def. The homotopy category $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$ has:

- objects: objects of $\mathcal{M}$.
- maps: $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X,Y) := \mathrm{Hom}_{\mathcal{M}}(RQX, RQY)/\sim$.

  (commuting R and Q is equivalent)
- $\mathrm{id}_X = [\mathrm{id}_{RQX}]$, $[g] \circ [f] = [g \circ f]$.

Canonical identity-on-objects functor $\gamma: \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ with $f \mapsto [RQf]$.

Thm. $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ is the localisation of $\mathcal{M}$ w.r.t. the weak equivalences.

Def. A functor $F : \mathcal{M} \to \mathcal{C}$ is homotopical if it sends w.e.'s to isos. A functor $F : \mathcal{M} \to \mathcal{N}$ is homotopical if it preserves w.e.'s. (difference: assume model categories to have (co)limits)

Ex. π_n, H_n, H^n (singular, complexes), not: (co)limits.

Lem. A homotopical functor $F : \mathcal{M} \to \mathcal{C}$ identifies left or right homotopic maps.

Proof. Let $H : \mathrm{Cyl}(X) \to Y$ be a (good) left homotopy from f to g. $i_0, i_1 : X \to \mathrm{Cyl}(X)$ both sections of cylinder projection $q : \mathrm{Cyl}(X) \xrightarrow{\sim} X$, so Fi_0, Fi_1 both sections of iso Fq $\rightsquigarrow Fi_0 = Fi_1$. Now: $Ff = FH \circ Fi_0 = FH \circ Fi_1 = Fg$. $\square$

Lem. A map f in $\mathcal{M}$ is a w.e. if and only if γf in $\mathrm{Ho}(\mathcal{M})$ is an iso.

Proof. f w.e. $\Leftrightarrow RQf$ w.e. (Q + R create w.e.)

$\Leftrightarrow RQf$ htpy equiv. (Whitehead)

$\Leftrightarrow [RQf] = \gamma f$ iso. $\square$

Proof (of thm). γ localisation: – γ homotopical ✓ (lemma)
– for any homotopical $F : \mathcal{M} \to \mathcal{C}$,

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{C} \\ \mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed, "\exists ! \mathrm{Ho}(F)"'] & \end{tikzcd} \qquad (*)$$

Define $\mathrm{Ho}(F) : X \mapsto F(X)$ on objects.
Have natural iso:

$$\alpha : F \underset{\cong}{\xRightarrow{(Fq)^{-1}}} FQ \underset{\cong}{\xRightarrow{FrQ}} FRQ.$$

Let $f : RQX \to RQY$ represent a map $X \to Y$ in $\mathrm{Ho}(\mathcal{M})$.
Then define

$$\mathrm{Ho}(F)([f]) : FX \xrightarrow{\alpha_X} FRQX \xrightarrow{Ff} FRQY \xrightarrow{\alpha_Y^{-1}} FY.$$

Well-defined: by lemma. (homotopical functor identifies htpic)
Functorial: by functoriality of F. $(\alpha_Y \circ \alpha_Y^{-1} = id)$
$(*)$ commutes: by naturality of α.
$(\alpha_Y^{-1} \circ FRQf \circ \alpha_X = \alpha_Y^{-1} \alpha_Y \circ Ff = Ff.)$

Uniqueness: if $f : RQX \to RQY$ represents a map $X \to Y$,
consider:

$$\begin{tikzcd} RQX \arrow[d, "f"] & QRQX \arrow[l, "q"', "\sim"] \arrow[r, "r", "\sim"'] \arrow[d, "Qf"] & RQRQX \arrow[d, "RQf"] \\ RQY & QRQY \arrow[l, "\sim", "q"'] \arrow[r, "\sim", "r"'] & RQRQY \end{tikzcd}$$

has image given by composite around diagram

become isos

must have image Ff by $(*)$

∎

<u>Cor.</u> Singular homology $H_n : \mathrm{Top} \to \mathrm{Ab}$ factors through $\mathrm{Ho}(\mathrm{Top})$. Homotopy groups $\pi_n : \mathrm{Top}_* \to \mathrm{Set}$ or Grp factor through $\mathrm{Ho}(\mathrm{Top}_*)$.

slice categories admit mod. str. created by $\mathcal{M}_{/x} \to \mathcal{M}$

<u>Cor.</u> γ induces an isomorphism

full subcat on homotopical functors

$$\gamma^* : \mathrm{Fun}(\mathrm{Ho}(\mathcal{M}), \mathcal{C}) \xrightarrow{\cong} \mathrm{Fun}^{\mathrm{ho}}(\mathcal{M}, \mathcal{C})$$

of categories.

Proof. Using similar techniques as thm. $\square$

<u>Rmk.</u> $\mathrm{Ho}(\mathcal{M}) \simeq h\mathcal{M}_{cf}$ with:

- objects: bifibrant objects of $\mathcal{M}$;
- htpy classes of maps in $\mathcal{M}$.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "RQ"] \arrow[dr, "\text{bij. obj.}"'] & \mathcal{M}_{cf} \arrow[r] & h\mathcal{M}_{cf} \\ & \mathrm{Ho}(\mathcal{M}) \arrow[ur, "\text{ff}"'] & \end{tikzcd}$$

Weaker univ. prop.: $\mathrm{Fun}(\mathrm{Ho}(\mathcal{M}), \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{ho}}(\mathcal{M}, \mathcal{C})$.

$\rightsquigarrow \mathrm{Ho}(\mathrm{Top}) \simeq \mathrm{Ho}(\mathrm{CW})$ of CW-complexes and htpy classes.
(by CW-approximation)

§ Derived functors

If $F: \mathcal{M} \to \mathcal{C}$ is homotopical, it factors through $\mathrm{Ho}(\mathcal{M})$ via $\mathrm{Ho}(F)$. What about non-homotopical functors?. We will consider approximations of $\mathrm{Ho}(F)$.
$\rightsquigarrow$ derived functors

For $F: \mathcal{M} \to \mathcal{N}$ homotopical between model categories, F does in general not factor through $\mathrm{Ho}(\mathcal{M})$ by thm., but $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\delta} \mathrm{Ho}(\mathcal{N})$ does!

Also consider approximations of $\mathrm{Ho}(\delta F)$ for non-homotopical F.
$\rightsquigarrow$ total derived functors

Will be used to compare model categories / homotopy theories.

Approximation problems occur more generally in category theory: studied using Kan extensions.

<u>Def.</u> A <u>left Kan extension</u> of $F : \mathcal{C} \to \mathcal{E}$ along $K : \mathcal{C} \to \mathcal{D}$ is:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow \eta"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, "\mathrm{Lan}_K F"'] &
\end{tikzcd}$$

Such that:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow \forall \alpha"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, "\forall G"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow \eta"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, bend left, "\mathrm{Lan}_K F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\Downarrow \exists !"] &
\end{tikzcd}$$

$\mathrm{Lan}_K F$ is <u>absolute</u> if for every $H : \mathcal{E} \to \mathcal{X}$, $H \circ \mathrm{Lan}_K F$ + $H\eta$ is a left Kan extension of HF along K.

Dually: a <u>right Kan extension</u> of F along K:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Uparrow \varepsilon"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, "\mathrm{Ran}_K F"'] &
\end{tikzcd}
\quad \text{s.t.} \quad
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Uparrow \forall \beta"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, "\forall G"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Uparrow \varepsilon"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, bend left] \arrow[ur, bend right] \arrow[ur, phantom, "\Uparrow \exists !"] &
\end{tikzcd}$$

Absoluteness: analogously.

Def. A left derived functor of $F:\mathcal{M}\to\mathcal{C}$ is an absolute right Kan extension of F along $\gamma:\mathcal{M}\to Ho(\mathcal{M})$, denoted $LF: Ho(\mathcal{M})\to\mathcal{C}$.
A right derived functor of F is an absolute left Kan extension of F along γ, denoted RF.

- UP $\Rightarrow$ uniqueness up to natural iso $\rightsquigarrow$ the derived functors.
- Already seen: homotopical functors $F \rightsquigarrow LF = RF = Ho\,F$.

Thm. If $F:\mathcal{M}\to\mathcal{C}$ takes w.e. between cofibrant objects to isos, then $LF := Ho(FQ)$ is a left derived functor of F.
Dually, if F takes w.e. between fibrant objects to isos, then $RF := Ho(FR)$ is a right derived functor of F.

Proof. FQ homotopical by assumption $\rightsquigarrow Ho(FQ)$.
Right Kan extension of F along γ:

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & {} \arrow[d, phantom, "\Uparrow Fq"] & \mathcal{C} \\
& Ho(\mathcal{M}) \arrow[ur, "Ho(FQ)"'] &
\end{tikzcd}$$

$Fq: FQ = Ho(FQ)\circ\gamma \Rightarrow F = F\,id.$

$$\begin{tikzcd} GQ \arrow[r, "\alpha Q"] \arrow[d, "Gq"'] & FQ \arrow[d, "Fq"] \\ G \arrow[r, "\alpha"'] & F \end{tikzcd}$$

Using $\mathrm{Fun}(\mathrm{Ho}(\mathcal{M}),\mathcal{C}) \simeq \mathrm{Fun}^{ho}(\mathcal{M},\mathcal{C})$, consider $G:\mathcal{M}\to\mathcal{C}$ homotopical and $\alpha: G \Rightarrow F$. By naturality, α factors as

$$\alpha : G \xRightarrow{(Gq)^{-1}} GQ \xRightarrow{\alpha Q} FQ \xRightarrow{Fq} F$$

(Gq iso since G homotopical, $Fq \circ \alpha Q \circ (Gq)^{-1} = \alpha \circ Gq \circ (Gq)^{-1} = \alpha$.)

Uniqueness: Suppose α factors as

$$\alpha : G \xRightarrow{\beta} FQ \xRightarrow{Fq} F.$$

By assumption on F, $FqQ : FQ^2 \Rightarrow FQ$ is an iso, so $\alpha Q \circ (Gq)^{-1}$ and β agree on cofibrant replacements. Naturality of β:

(G and FQ homotopical)

$$\begin{tikzcd} GQ \arrow[r, Rightarrow, "\beta Q"] \arrow[d, Rightarrow, "Gq"', "\cong"] & FQ^2 \arrow[d, Rightarrow, "FQq", "\cong"'] \\ G \arrow[r, Rightarrow, "\beta"'] & FQ \end{tikzcd}$$

So β is determined by βQ.

Absoluteness: if $H : \mathcal{C} \to \mathcal{D}$ is any functor, then HFQ is also homotopical. Hence:

$$H \circ \mathrm{Ho}(FQ) \circ \gamma = HFQ = \mathrm{Ho}(HFQ) \circ \gamma \rightsquigarrow H \circ \mathrm{Ho}(FQ) = \mathrm{Ho}(HFQ).$$

(UP of localisation)

Argument above shows that

$$(\mathrm{Ho}(HFQ) = H \circ \mathrm{Ho}(FQ),\ HFq)$$

is a right Kan extension of HF along γ. $\square$

<u>Def.</u> A <u>total left derived functor</u> of $F: \mathcal{M} \to \mathcal{N}$ is a left derived functor $\mathbb{L}F: \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ of the composite $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\delta} \mathrm{Ho}(\mathcal{N})$:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \arrow[d, "\delta"] \\ \mathrm{Ho}(\mathcal{M}) \arrow[r, "\mathbb{L}F"'] & \mathrm{Ho}(\mathcal{N}) \end{tikzcd}$$

<u>Cor.</u> If $F: \mathcal{M} \to \mathcal{N}$ takes w.e.'s between cofibrant objects in $\mathcal{M}$ to w.e.'s in $\mathcal{N}$, then $\mathbb{L}F := \mathrm{Ho}(\delta F Q)$ is a total left derived functor of F

apply levelwise

<u>Ex.</u> $F: \mathrm{Mod}_R \to \mathrm{Mod}_S$ additive $\rightsquigarrow F: Ch_{\geq 0}(S) \to Ch_{\geq 0}(R)$ preserving chain homotopies. Quasi-isos between complexes of projectives are chain homotopies, so F takes w.e. between fibrant complexes (w.r.t. proj. mod. str.) to w.e.

$$\begin{tikzcd} \mathbb{L}F: \mathcal{D}_{\geq 0}(\mathrm{Mod}_R) \arrow[r] & \mathcal{D}_{\geq 0}(\mathrm{Mod}_S) \arrow[d, "H_i"] \\ \mathrm{Mod}_R \arrow[u, "\deg 0"] \arrow[r, dashed, "L_iF"'] & \mathrm{Mod}_S \end{tikzcd}$$

$\rightsquigarrow$ by projective resolution and F

(right exactness assumed to get the l.e.s)

Often derived functors arise from adjunctions.

<u>Def.</u> An adjunction $\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "U"] \end{tikzcd}$ is a <u>Quillen adjunction</u> if F preserves cofibrations and acyclic cofibrations.

<u>Lem</u>. For $\mathcal{M} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{N}$ ($F \dashv U$), TFAE:

1. F preserves cof. + ac. cof.
2. U ——— fib. + ac. fib.
3. F preserves cof + U preserves fib.
4. ——— ac. cof. + ——— ac. fib.

This follows from:

<u>Lem</u>. $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$ ($F \dashv U$) adjunction. Then $Fi \boxslash p \iff i \boxslash Up$.

Proof (sketch).

$$\begin{tikzcd} FA \arrow[r] \arrow[d, "Fi"'] & X \arrow[d, "p"] \\ FB \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & UX \arrow[d, "Up"] \\ B \arrow[r] \arrow[ur, dashed] & UY \end{tikzcd}$$

$\square$

Lem (Ken Brown). If $F : \mathcal{M} \to \mathcal{N}$ takes ac. cof. between cofibrant objects to w.e.'s, then F takes all w.e.'s between cofibrant objects to w.e.'s.

Proof. Let $f : X \xrightarrow{\sim} Y$ be a w.e. between cof. objects: Factor $(f, \mathrm{id}_Y) : X \amalg Y \to Y$:

(cograph)

$$\begin{tikzcd}
 & X \arrow[dr, tail, "j_X"] \arrow[drr, bend left, "\sim"] \arrow[drrr, bend left=40, "f\ \sim"] & & & \\
\emptyset \arrow[ur, tail] \arrow[dr, tail] & & X \amalg Y \arrow[r, tail, "i"] & Z \arrow[r, two heads, "p", "\sim"'] & Y \\
 & Y \arrow[ur, tail, "j_Y"'] \arrow[urr, bend right, "\sim"'] \arrow[urrr, bend right=40, "="'] & & &
\end{tikzcd}$$

pushout

composition

Z cofibrant by $\emptyset \rightarrowtail X \rightarrowtail Z$

Apply F:
$$\begin{tikzcd}
FX \arrow[dr, "Fj_X", "\sim"'] \arrow[drr, bend left, "Ff"] & & \\
 & FZ \arrow[r, "Fp"] & FY \\
FY \arrow[ur, "Fj_Y", "\sim"'] \arrow[urr, bend right, "="'] & &
\end{tikzcd}$$
$\overset{2/3}{\Longrightarrow}$ Ff w.e.

$\square$

Thm. $\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "U"] \end{tikzcd}$ Quillen adjunction. Then the total derived functors exist and form an adjunction

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}) \arrow[r, bend left, "\mathbb{L}F"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathcal{N}) \arrow[l, bend left, "\mathbb{R}U"] \end{tikzcd}$$

Proof due to Georges Maltsiniotis, 2007

Proof. Existence: Ken Brown's lemma and earlier results.

$\eta : \mathrm{id}_{\mathcal{M}} \Rightarrow UF$, $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{N}}$ unit and counit,

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Uparrow\lambda"] & \mathcal{N} \arrow[d, "\delta"] \\ \mathrm{Ho}(\mathcal{M}) \arrow[r, "\mathbb{L}F"'] & \mathrm{Ho}(\mathcal{N}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{N} \arrow[r, "U"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Downarrow\rho"] & \mathcal{M} \arrow[d, "\gamma"] \\ \mathrm{Ho}(\mathcal{N}) \arrow[r, "\mathbb{R}U"'] & \mathrm{Ho}(\mathcal{M}) \end{tikzcd}$$

$\mathbb{L}F$ abs. right Kan ext. of δF along γ

$\rightsquigarrow$ $\mathbb{R}U \circ \mathbb{L}F$ right Kan ext of $\mathbb{R}U\delta F$ along γ

Natural transformation

$$\gamma \xRightarrow{\gamma\eta} \gamma UF \xRightarrow{\rho F} \mathbb{R}U \circ \delta F$$

induces derived unit $\tilde{\eta}$:

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\mathbb{R}U\circ\delta F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\Uparrow \rho F \circ \gamma\eta"] & \mathrm{Ho}(\mathcal{M}) \\ & \mathrm{Ho}(\mathcal{M}) \arrow[ur, "="'] & \end{tikzcd} = \begin{tikzcd} \mathcal{M} \arrow[rr, "\mathbb{R}U\circ\delta F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\Uparrow \mathbb{R}U\lambda"] & \mathrm{Ho}(\mathcal{M}) \\ & \mathrm{Ho}(\mathcal{M}) \arrow[ur, bend left, "\mathbb{R}U\circ\mathbb{L}F"] \arrow[ur, bend right, ""'] \arrow[ur, phantom, "\Uparrow\exists!\tilde{\eta}"'] & \end{tikzcd}$$

↝ Obtain derived unit $\tilde{\eta} : \mathrm{id}_{\mathrm{Ho}(\mathcal{M})} \Rightarrow \mathbb{R}U \circ \mathbb{L}F$ and counit $\tilde{\varepsilon} : \mathbb{L}F \circ \mathbb{R}U \Rightarrow \mathrm{id}_{\mathrm{Ho}(\mathcal{N})}$ satisfying:

$$\mathbb{R}U\lambda \circ \tilde{\eta}\gamma = \rho F \circ \gamma\eta, \qquad \tilde{\varepsilon}\delta \circ \mathbb{L}F\rho = \delta\varepsilon \circ \lambda U. \quad (*)$$

Triangle identities: from UP of $\mathbb{L}F$ and $\mathbb{R}U$ (uniqueness to see something is the identity), $(*)$ and symbol pushing. (naturality $\lambda, \rho, \tilde{\eta}, \tilde{\varepsilon}$; triangle id's $F \dashv U$) $\square$

Model categorical criterion for when a Quillen adjunction is an equivalence.

<u>Def.</u> A Quillen adjunction $\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "U"] \end{tikzcd}$ is a <u>Quillen equivalence</u> if for all cofibrant A in $\mathcal{M}$ and all fibrant X in $\mathcal{N}$,

$$f^{\#} : FA \to X \text{ w.e.} \iff f^{\flat} : A \to UX \text{ w.e.}$$

<u>Prop.</u> $\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "U"] \end{tikzcd}$ Quillen adjunction. Then $F \dashv U$ is a Quillen equivalence if and only if the derived adjunction $\mathbb{L}F \dashv \mathbb{R}U$ is an adjoint equivalence.

§ Examples

① Spaces

Thm. $sSet_{Kan} \underset{Sing}{\overset{|-|}{\rightleftarrows}} Top_{Quillen}$ is a Quillen equivalence.

$\uparrow$ = fundamental ∞-groupoid

② $(\infty,1)$-categories

Thm. $sSet_{Joyal} \underset{\tilde{N}}{\overset{\mathfrak{C}}{\rightleftarrows}} sCat_{Bergner}$ is a Quillen equivalence.

$\mathfrak{C}$ ← rigidification

$\tilde{N}$ ← homotopy-coherent nerve

③ Equivariant homotopy theory (G finite group)

Thm (Elmendorf). $Fun(Orb_G^{op}, Top)_{proj} \underset{}{\overset{ev_{G/e}}{\rightleftarrows}} G\text{-}Top_{f.p.}$ is a Q.E.

w.e. + fib. pointwise

w.e. + fib. created by fixed points $\forall H \leq G$

④ Dold-Kan

Thm (Schwede-Shipley). $Ch_{\geq 0}(\mathbb{Z})_{proj} \underset{N}{\overset{\Gamma}{\rightleftarrows}} sAb_{Kan}$ is a Q.E.

equiv of 1-cats, already an so either one is left/right adj.