

Homotopy category + Quillen functors

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- Today: Previously: axioms, examples, homotopy relation
- Homotopy category $\text{Ho}(\mathcal{M})$: localisation of model category \mathcal{M} w.r.t. weak equivalences
 - Derived functors between homotopy categories

Throughout: \mathcal{M}, \mathcal{N} model categories
 \mathcal{C}, \mathcal{D} categories

§ (Co)fibrant replacement

Assume: model categories have functorial factorisations.

Okay: most mod. cat's we care about have this, by small obj. arg.

Def. A functorial factorisation on \mathcal{C} is a section of $d_1 = \circ : \text{Fun}([2], \mathcal{C}) \rightarrow \text{Fun}([1], \mathcal{C})$.

Cor. A model category \mathcal{M} has an endofunctor Q with a natural w.e. $q : Q \xrightarrow{\cong} \text{id}_{\mathcal{M}}$ s.t. QX is cofibrant for every $X \in \mathcal{M}$. We call Q cofibrant replacement.

Dually: fibrant replacement $r : \text{id}_{\mathcal{M}} \xrightarrow{\cong} R$.

Proof. Factor $\begin{array}{ccc} \emptyset & \rightarrow & X \\ & \searrow & \nearrow \\ & & QX \end{array}$ or $\begin{array}{ccc} X & \rightarrow & * \\ & \searrow & \nearrow \\ & & RX \end{array}$

□

Lemma. Q and R create weak equivalences.

Proof. $f: X \rightarrow Y$ in $\mathcal{M} \rightsquigarrow QX \xrightarrow{q_x} X$

$$\begin{array}{ccc} Qf \downarrow & & \downarrow f \\ QY & \xrightarrow[\sim]{q_y} & Y \end{array}$$

□

§ Homotopy category

Recall: Homotopy relation \sim on $\text{Hom}_{\mathcal{M}}(A, X)$ equiv. w.r. to A and X fib., respected by composition.

Def. The homotopy category $\text{Ho}(\mathcal{M})$ of \mathcal{M} has:

- objects: objects of \mathcal{M} .

- maps: $\text{Hom}_{\text{Ho}(\mathcal{M})}(X, Y) := \text{Hom}_{\mathcal{M}}(RQX, RQY) / \sim$.

(commuting R and Q is equivalent)

- $\text{id}_X = [id_{RQX}]$, $[q] \circ [f] = [q \circ f]$.

Canonical identity-on-objects functor $\gamma: \mathcal{M} \rightarrow \text{Ho}(\mathcal{M})$ with $f \mapsto [RQf]$.

Thm. $\gamma: \mathcal{M} \rightarrow \text{Ho}(\mathcal{M})$ is the localisation of \mathcal{M} w.r.t. the weak equivalences.

Def. A functor $F: \mathcal{M} \rightarrow \mathcal{C}$ is homotopical if it sends w.e.s to isos. A functor $F: \mathcal{M} \rightarrow \mathcal{N}$ is homotopical if it preserves w.e.s. (Difference: assume model categories to have (co)limits)

Ex. $\mathcal{U}_n, \mathcal{K}_n, \mathcal{M}^n$ (singular, complexes), not: (co)limits.

Lem. A homotopical functor $F: \mathcal{M} \rightarrow \mathcal{C}$ identifies left or right homotopic maps. (good)

Proof. Let $H: \text{Cyl}(X) \rightarrow Y$ be a left homotopy from f to g .
 $i_0, i_1: X \rightarrow \text{Cyl}(X)$ both sections of cylinder projection
 $q: \text{Cyl}(X) \xrightarrow{\sim} X$, so $F i_0, F i_1$ both sections of iso $F q$
 $\leadsto F i_0 = F i_1$. Now: $F f = F H \circ F i_0 = F H \circ F i_1 = F g$. \square

Lem. A map f in \mathcal{M} is a w.e. if and only if γf in $\text{Ho}(\mathcal{M})$ is an iso.

Proof. f w.e. $\Leftrightarrow \mathcal{R}Qf$ w.e. ($\mathcal{Q} + \mathcal{R}$ create w.e.)
 $\Leftrightarrow \mathcal{R}Qf$ htq equiv. (Whitehead)
 $\Leftrightarrow [\mathcal{R}Qf] = \gamma f$ iso. \square

Proof (of thm). γ localisation: - γ homotopical \checkmark (lemma)
 - for any homotopical $F: \mathcal{M} \rightarrow \mathcal{C}$,

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{C} \\ \gamma \downarrow & \dashrightarrow & \uparrow \\ \text{Ho}(\mathcal{M}) & \xrightarrow{\exists! \text{Ho}(F)} & \end{array} \quad (*)$$

Define $\text{Ho}(F): X \mapsto F(X)$ on objects.

Have natural iso:

$$\alpha: F \xrightarrow[\cong]{(Fq)^{-1}} Fq \xrightarrow[\cong]{Fr} FqQ$$

Let $f: QqX \rightarrow QqY$ represent a map $X \rightarrow Y$ in $\text{Ho}(\mathcal{M})$.

Then define

$$\text{Ho}(F)([f]) : Fx \xrightarrow{\alpha_x} FQqX \xrightarrow{Ff} FQqY \xrightarrow{\alpha_y^{-1}} Fy$$

Well-defined: by lemma. (homotopical functor identifies homotid)

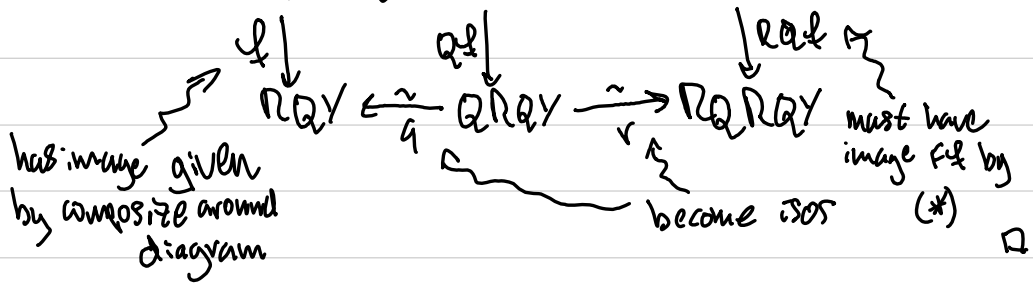
Functorial: by functoriality of F . ($\alpha_y \circ \alpha_x^{-1} = \text{id}$)

(*) commutes: by naturality of α .

$$(\alpha_y^{-1} \circ FQq \circ \alpha_x = \alpha_y^{-1} \circ \alpha_x \circ Ff = Ff)$$

Uniqueness: if $f: QqX \rightarrow QqY$ represents a map $X \rightarrow Y$,

consider: $QqX \xleftarrow{q} QqQqX \xrightarrow{r} QqQqX$



Cor. Singular homology $H_n: \text{Top} \rightarrow \text{Ab}$ factors through $\text{Ho}(\text{Top})$. Homotopy groups $\pi_n: \text{Top}_* \rightarrow \text{Set}$ or Grp factor through $\text{Ho}(\text{Top}_*)$.

↗ slice categories admit mod. str.

created by $\mathcal{M}/x \rightarrow \mathcal{M}$

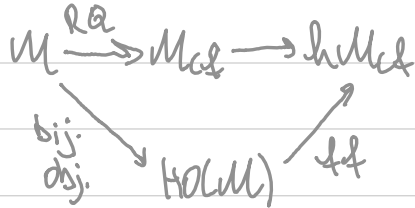
Cor. γ induces an isomorphism
 $\gamma^*: \text{Fun}(\text{Ho}(\mathcal{M}), \mathcal{C}) \xrightarrow{\cong} \text{Fun}^{\text{ho}}(\mathcal{M}, \mathcal{C})$
 of categories.

full subcat
 of homotopical functors

Proof. Using similar techniques as thm. □

Rmk. $\text{Ho}(\mathcal{M}) \simeq \text{hull}_{\mathcal{C}}$ with:

- objects: bifibrant objects of \mathcal{M} ;
- htpy classes of maps in \mathcal{M} .



Weaker univ. prop.: $\text{Fun}(\text{Ho}(\mathcal{M}), \mathcal{C}) \simeq \text{Fun}^{\text{ho}}(\mathcal{M}, \mathcal{C})$.

↪ $\text{Ho}(\text{Top}) \simeq \text{Ho}(\mathcal{C}w)$ of $\mathcal{C}w$ -complexes and htpy classes.
 (by $\mathcal{C}w$ -approximation)

§ Derived functors

If $F: \mathcal{M} \rightarrow \mathcal{E}$ is homotopical, it factors through $\mathrm{Ho}(\mathcal{M})$ via $\mathrm{Ho}(F)$. What about non-homotopical functors? We will consider approximations of $\mathrm{Ho}(F)$.

→ derived functors

For $F: \mathcal{M} \rightarrow \mathcal{N}$ homotopical between model categories, F does in general not factor through $\mathrm{Ho}(\mathcal{M})$ by Thm., but $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\delta} \mathrm{Ho}(\mathcal{N})$ does!

Also consider approximations of $\mathrm{Ho}(SF)$ for non-homotopical F .

→ total derived functors

Will be used to compare model categories / homotopy theories.

Approximation problems occur more generally in category theory: Studied using Kan extensions.

Def. A left Kan extension of $F: \mathcal{C} \rightarrow \mathcal{E}$ along $K: \mathcal{C} \rightarrow \mathcal{D}$ is:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ & \searrow K & \downarrow \eta \\ & & \mathcal{D} \end{array} \quad \begin{array}{c} \downarrow \eta \\ \nearrow \text{Lan}_K F \end{array}$$

Such that:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ & \searrow K & \downarrow \eta \\ & & \mathcal{D} \end{array} \quad \begin{array}{c} \downarrow \eta \\ \nearrow \text{Lan}_K F \end{array} = \begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ & \searrow K & \downarrow \eta \\ & & \mathcal{D} \end{array} \quad \begin{array}{c} \downarrow \eta \\ \nearrow \text{Lan}_K F \\ \downarrow \eta \\ \nearrow G \end{array}$$

$\text{Lan}_K F$ is absolute if for every $K: \mathcal{C} \rightarrow \mathcal{D}$, $H \circ \text{Lan}_K F + H\eta$ is a left Kan extension of HF along K .

Dually: a right Kan extension of F along K :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ & \nearrow K & \downarrow \eta \\ & & \mathcal{D} \end{array} \quad \begin{array}{c} \uparrow \epsilon \\ \searrow \text{Ran}_K F \end{array} \quad \text{s.t.} \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ & \nearrow K & \downarrow \eta \\ & & \mathcal{D} \end{array} \quad \begin{array}{c} \uparrow \epsilon \\ \searrow \text{Ran}_K F \\ \uparrow \epsilon \\ \searrow G \end{array} = \begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ & \nearrow K & \downarrow \eta \\ & & \mathcal{D} \end{array} \quad \begin{array}{c} \uparrow \epsilon \\ \searrow \text{Ran}_K F \\ \uparrow \epsilon \\ \searrow G \end{array}$$

Absoluteness: analogously.

Def. A left derived functor of $F: \mathcal{M} \rightarrow \mathcal{C}$ is an absolute right Kan extension of F along $\gamma: \mathcal{M} \rightarrow \text{Ho}(\mathcal{M})$, denoted $LF: \text{Ho}(\mathcal{M}) \rightarrow \mathcal{C}$.

A right derived functor of F is an absolute left Kan extension of F along γ , denoted RF .

- $\text{UP} \Rightarrow$ uniqueness up to natural iso \leadsto the derived functors.
- Already seen: homotopical functors $F \leadsto LF = RF = \text{Ho} F$.

Thm. If $F: \mathcal{M} \rightarrow \mathcal{C}$ takes w.e. between cofibrant objects to Kos, then $LF := \text{Ho}(FQ)$ is a left derived functor of F .

Dually, if F takes w.e. between fibrant objects to Kos, then $RF := \text{Ho}(FR)$ is a right derived functor of F .

Proof. FQ homotopical by assumption $\leadsto \text{Ho}(FQ)$.

Right Kan extension of F along γ :

$$\begin{array}{ccc}
 \mathcal{M} & \xrightarrow{F} & \mathcal{C} \\
 \searrow \gamma & \Uparrow FQ & \nearrow \text{Ho}(FQ) \\
 & \text{Ho}(\mathcal{M}) &
 \end{array}$$

$$Fq: FQ = \text{Ho}(FQ) \circ \gamma \Rightarrow F = \text{Fid.}$$

$$\begin{array}{ccc}
 GQ & \xrightarrow{\alpha Q} & FQ \\
 Gq \downarrow & & \downarrow Fq \\
 G & \xrightarrow{\alpha} & F
 \end{array}$$

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Using $\text{Fun}(\text{Ho}(\mathcal{M}), \mathcal{C}) \simeq \text{Fun}^{\text{ho}}(\mathcal{M}, \mathcal{C})$, consider $G: \mathcal{M} \rightarrow \mathcal{C}$ homotopical and $\alpha: G \Rightarrow F$. By naturality, α factors as

$$\alpha: G \xrightarrow{(Gq)^{-1}} GQ \xrightarrow{\alpha Q} FQ \xrightarrow{Fq} F$$

(Gq iso since G homotopical, $Fq \circ \alpha Q \circ (Gq)^{-1} = \alpha \circ Gq \circ (Gq)^{-1} = \alpha$.)

Uniqueness: Suppose α factors as

$$\alpha: G \xrightarrow{\beta} FQ \xrightarrow{Fq} F.$$

By assumption on F , $Fq: FQ \rightarrow F$ is an iso, so $\alpha Q \circ (Gq)^{-1}$ and β agree on cofibrant replacements. Naturalizy of β :

$$\begin{array}{ccc}
 GQ & \xrightarrow{\beta Q} & FQ^2 \\
 Gq \downarrow \cong & & \cong \downarrow Fq \\
 G & \xrightarrow{\beta} & Fq
 \end{array}$$

(G and Fq homotopical)

So β is determined by βQ .

Absoluteness: if $H: \mathcal{C} \rightarrow \mathcal{D}$ is any functor, then $H FQ$ is also homotopical. Hence:

$$H \circ \text{Ho}(Fq) \circ \gamma = H FQ = \text{Ho}(H FQ) \circ \gamma \rightsquigarrow H \circ \text{Ho}(Fq) = \text{Ho}(H FQ).$$

Argument above shows that

$$(\text{Ho}(H FQ)) = H \circ \text{Ho}(Fq), H Fq$$

is a right Kan extension of $H F$ along γ .

□

Def. A total left derived functor of $F: \mathcal{M} \rightarrow \mathcal{N}$ is a left derived functor $\mathbb{L}F: \text{Ho}(\mathcal{M}) \rightarrow \text{Ho}(\mathcal{N})$ of the composite

$$\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\delta} \text{Ho}(\mathcal{N}): \quad \begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ \gamma \downarrow & & \downarrow \delta \\ \text{Ho}(\mathcal{M}) & \xrightarrow{\mathbb{L}F} & \text{Ho}(\mathcal{N}) \end{array}$$

Cor. 1.1 If $F: \mathcal{M} \rightarrow \mathcal{N}$ takes w.e.s between abtriviant objects in \mathcal{M} to w.e.s in \mathcal{N} , then $\mathbb{L}F := \text{Ho}(\delta \circ F \circ \gamma)$ is a total left derived functor of F

apply likewise

Ex. $F: \text{Mod}_R \rightarrow \text{Mod}_S$ additive $\rightsquigarrow F: \text{Ch}_{\geq 0}(S) \rightarrow \text{Ch}_{\geq 0}(R)$ preserving chain homotopies. Quasi-isos between complexes of projectives are chain homotopies, so F takes w.e. between fibrant complexes (w.r.t. proj. mod. str.) to w.e.

$\rightsquigarrow \mathbb{L}F: \mathcal{D}_{\geq 0}(\text{Mod}_R) \rightarrow \mathcal{D}_{\geq 0}(\text{Mod}_S)$ by projective resolution and F

$$\begin{array}{ccc} \text{Mod}_R & \xrightarrow{\mathbb{L}F} & \text{Mod}_S \\ \text{des}_0 \uparrow & & \downarrow M_i \\ \text{Mod}_R & \dashrightarrow & \text{Mod}_S \\ & \text{LiF} & \end{array}$$

(right exactness assumed to get the l.e.s)

Other derived functors arise from adjunctions.

Def. An adjunction $\mathcal{M} \xrightleftharpoons[\perp]{F} \mathcal{N}$ is a Quillen adjunction if F preserves cofibrations and acyclic cofibrations.

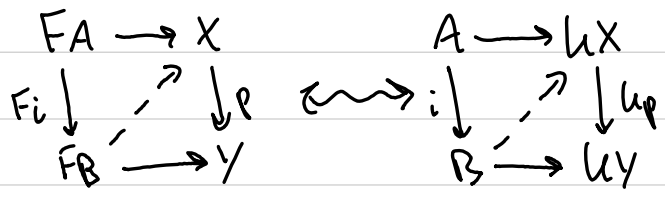
Lem. For $\mathcal{M} \xrightleftharpoons[u]{F} \mathcal{N}$, \perp FAE:

- ① F preserves cof. + ac. cof.
- ② u — fib. + ac. fib.
- ③ F preserves cof + u preserves fib.
- ④ — ac. cof. + — ac. fib.

This follows from:

Lem. $\mathcal{C} \xrightleftharpoons[u]{F} \mathcal{D}$ adjunction. Then $F_i \dashv p \Leftrightarrow i \dashv u_p$.

Proof (sketch).



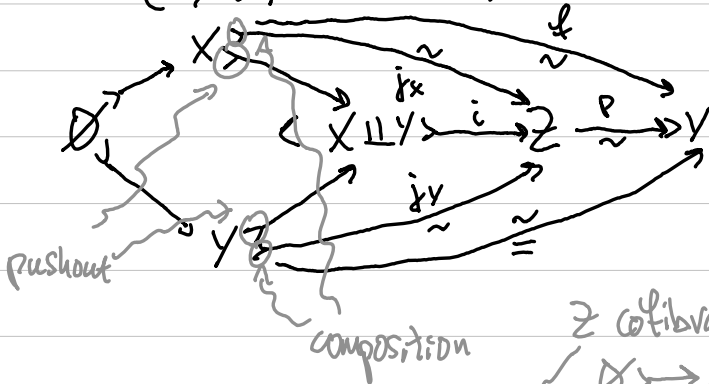
□

Lem (Ken Brown). If $F: \mathcal{M} \rightarrow \mathcal{W}$ takes co. cot. between cofibrant objects to w.e.'s, then F takes all w.e.'s between cofibrant objects to w.e.'s.

Proof. Let $f: X \xrightarrow{\sim} Y$ be a w.e. between cof. objects:

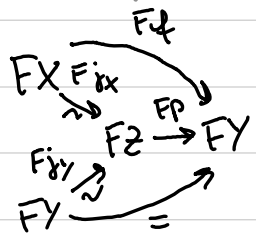
Factor $(f, id_Y): X \amalg Y \rightarrow Y$:

(Cograph)



Z cofibrant by $D \rightarrow X \rightarrow Z$

Apply F :



$\frac{2/3}{\implies} Ff$ w.e.

□

Thm. $\mathcal{M} \xrightleftharpoons[u]{F} \mathcal{N}$ Quillen adjunction. Then the total derived functors exist and form an adjunction

$$\text{Ho}(\mathcal{M}) \xrightleftharpoons[\mathbb{R}u]{\mathbb{L}F} \text{Ho}(\mathcal{N})$$

Proof due to Georges Matisimiotis, 2007

Proof. Existence: Kan Brown's lemma and earlier results.

$\eta: id_{\mathcal{M}} \Rightarrow uF$, $\epsilon: Fu \Rightarrow id_{\mathcal{N}}$ unit and counit,

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} & & \mathcal{N} & \xrightarrow{u} & \mathcal{M} \\ \delta \downarrow & \uparrow \eta & \downarrow \epsilon & & \delta \downarrow & \downarrow \epsilon & \downarrow \delta \\ \text{Ho}(\mathcal{M}) & \xrightarrow{\mathbb{L}F} & \text{Ho}(\mathcal{N}) & & \text{Ho}(\mathcal{N}) & \xrightarrow{\mathbb{R}u} & \text{Ho}(\mathcal{M}) \end{array}$$

$\mathbb{L}F$ abs. right Kan ext. of F along δ

$\Rightarrow \mathbb{R}u \circ \mathbb{L}F$ right Kan ext of uF along δ

Natural transformation

$$\delta \xrightarrow{\gamma \eta} \delta uF \xrightarrow{PF} \mathbb{R}u \circ \delta F$$

induces derived unit $\tilde{\eta}$:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathbb{R}u \circ \delta F} & \text{Ho}(\mathcal{M}) \\ \delta \downarrow & \uparrow \mathbb{R}PF \circ \gamma \eta & \uparrow = \\ \text{Ho}(\mathcal{M}) & & \end{array}$$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathbb{R}u \circ \delta F} & \text{Ho}(\mathcal{M}) \\ \delta \downarrow & \uparrow \mathbb{R}u \circ \delta F & \uparrow \tilde{\eta} \\ \text{Ho}(\mathcal{M}) & & \end{array}$$

↪

Obtain derived unit $\tilde{\eta}: \text{id}_{\text{Mod}} \Rightarrow \mathbb{R}u \circ \mathbb{L}F$ and counit

$\tilde{\epsilon}: \mathbb{L}F \circ \mathbb{R}u \Rightarrow \text{id}_{\text{Mod}}$ satisfying:

$$\mathbb{R}u \circ \tilde{\eta} \circ \gamma = \rho \circ \gamma, \quad \tilde{\epsilon} \circ \delta \circ \mathbb{L}F = \delta \circ \epsilon \circ \lambda. \quad \text{(*)}$$

Triangle identities: from \mathbb{R} of $\mathbb{L}F$ and $\mathbb{R}u$ (uniqueness to see something is the identity), ~~(*)~~ and symbol pushing. (naturality $\lambda, \rho, \tilde{\eta}, \tilde{\epsilon}$; triangle id's F-U) \square

Model categorical criterion for when a Quillen adjunction is an equivalence.

Def. A Quillen adjunction $\mathcal{M} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{u} \end{array} \mathcal{W}$ is a Quillen equivalence if for all cofibrant A in \mathcal{M} and all fibrant X in \mathcal{W} , $f^\#: FA \rightarrow X$ w.e. \iff $f^\flat: A \rightarrow uX$ w.e.

Prop. $\mathcal{M} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{u} \end{array} \mathcal{W}$ Quillen adjunction. Then $F \dashv u$ is a Quillen equivalence if and only if the derived adjunction $\mathbb{L}F \dashv \mathbb{R}u$ is an adjoint equivalence.

§ Examples

① Spaces

Thm. $\text{sSet}_{\text{Kan}} \xrightleftharpoons[\text{Sing}]{\text{H}}$ $\text{Top}_{\text{Quillen}}$ is a Quillen Equivalence.
 = fundamental ∞ -groupoid

② $(\infty, 1)$ -categories

Thm. $\text{sSet}_{\text{Joyal}} \xrightleftharpoons[\tilde{N}]{\mathcal{C}}$ $\text{sCat}_{\text{Reyner}}$ is a Quillen Equivalence.
 \mathcal{C} = rigidification
 \tilde{N} = homotopy-coherent nerve

③ Equivariant homotopy theory (G finite group)

Thm (Eilenberg). $\text{Fun}(\text{Orb}_G^{\text{op}}, \text{Top}) \xrightleftharpoons[\text{Proj}]{\text{Eva/E}}$ $G\text{-Top}_{\text{f.p.}}$ is a Q.E.
 w.e. + fib. pairwise
 w.e. + fib. created by fixed points $\forall H \leq G$

④ Dold-Kan

Thm (Schwede-Simplicial). $\text{Ch}_{\geq 0}(\mathbb{Z}) \xrightleftharpoons[\text{Proj}]{\Gamma}$ SAb_{Kan} is a Q.E.
 already an equiv at 1-cat, so either one is left/right adj.